SOME GEOMETRIC FACETS OF THE LANGLANDS CORRESPONDENCE FOR REAL GROUPS

0.1. **Conventions.** Throughout 'variety' = 'separated scheme of finite type over Spec(k)', where k is either the field of complex numbers $\mathbf{C}$, or the algebraic closure $\bar{\mathbf{F}}_q$ of a finite field $\mathbf{F}_q$. A point of a variety will always mean a geometric point, and the notation '$x \in X$', for a variety X, will be used in lieu of 'let x be a geometric point of X'. A *fibration* will mean a morphism of varieties which is locally trivial (on the base) in the ètale sense.

For varieties over Spec$(\mathbf{C})$, sheaves, cohomology, etc. will always be with $\mathbf{R}$ or $\mathbf{C}$ coefficients, and with respect to the complex analytic site. For varieties over Spec$(\bar{\mathbf{F}}_q)$, sheaves should be understood to be ℓ-adic sheaves with ℓ a prime number different from char$(\mathbf{F}_q)$. 'Local system' in the latter setting should be understood to be a smooth ℓ-adic sheaf.

Given an algebraic group G, we write G^0 for its identity component. Suppose G acts on X. Then we write G_x for the isotropy group of a point $x \in X$. Given a principle G-fibration $E \to B$, we write $E \times^G X \to B$ for the associated fibration.[1] We will also assume that

for each $x \in X$, the orbit map $G \to X$, $g \mapsto g \cdot x$, is separable.

The assumption implies that an orbit $G \cdot x$ is isomorphic to G/G_x as a G-variety. This is always satisfied in characteristic 0. In the situations we will be interested in below it is also always satisfied for characteristic $\neq 2$ [MS, §3.1.1].

We write $D_G(X)$ for the G-equivariant derived category (in the sense of [BL]), and $\mathrm{Perv}_G(X) \subseteq D_G(X)$ for the abelian subcategory of equivariant perverse sheaves on X. Perverse cohomology is denoted by ${}^pH^*$. Change of group functors (restriction, induction equivalence, quotient equivalence, etc.) will often be omitted from the notation. All functors between derived categories will be tacitly derived. Both the functor of G-equivariant cohomology as well as the G-equivariant cohomology ring of a point will be denoted by H^*_G.

0.2. $\mathbf{B}\backslash\mathbf{G}/\mathbf{K}$**.** Let G be a connected reductive group, $\theta\colon G \to G$ a non-trivial involution, T a θ-stable maximal torus, and $B \supseteq T$ a θ-stable Borel containing it (such a pair (B, T) always exists, see [St, §7]). When we are working over the algebraic closure of a finite field $\mathbf{F}_q$, we assume that q is large enough that G, B, θ are defined over $\mathbf{F}_q$, T is $\mathbf{F}_q$-split, etc.

Write W for the Weyl group. Let $K = G^\theta$ denote the fixed point subgroup. Then

(i) K is reductive (but not necessarily connected, see [V, §1]);
(ii) $|B\backslash G/K| < \infty$ (a convenient reference is [MS, §6]);
(iii) K-orbits in G/B are affinely embedded (see [M, Ch. H, Proposition 1]);

[1] Generally, $E \times^G X$ is only an algebraic space. It is a variety if, for instance, X is quasi-projective with linearized G-action; or G is connected and X can be equivariantly embedded in a normal variety (Sumihiro's Theorem). One of these assumptions will always be satisfied below.

(iv) for each $x \in G/B$, the component group K_x/K_x^0 has exponent 2 [V, Proposition 7].

Our primary concern is the category $D_{B\times K}(G)$, for the $B \times K$-action given by $(b,k)\cdot g = bgk^{-1}$. The evident identification of $B \times K$-orbits in G, with B-orbits in G/K, and with K-orbits in G/B, respects closure relations. There are corresponding identifications: $D_B(G/K) = D_{B\times K}(G) = D_K(G/B)$. We will use these identifications without further comment.

Let $s \in W$ be a simple reflection, $P \supseteq B$ the corresponding minimal parabolic, and v a B-orbit in G/K. Then the subvariety $P \cdot v \subseteq G/K$ contains a unique open dense B-orbit $s \star v$. Let $\leq$ denote the closure order on orbits, i.e., $v \leq w$ if and only if v is contained in the closure $\overline{w}$.

Theorem 0.2.1 ([RS, Theorem 7.11]). *The order $\leq$ is the weakest partial order on $B\backslash G/K$ such that $v \leq s \star v$ for each simple reflection s.*

Corollary 0.2.2. *If $w \in B\backslash G/K$ is not closed, then there exists a simple reflection $s \in W$, and $v \in B\backslash G/K$ such that $v \lneq w$ and $s \star v = w$.*

Let $\pi\colon G/B \to G/P$ be the evident projection. Let $x \in G/B$. Set $y = \pi(x)$, and $L_y^s = \pi^{-1}(y)$. Note: $L_y^s \simeq \mathbf{P}^1$.

(∗)
$$\begin{array}{ccccc} \mathbf{P}^1 & \xrightarrow{\sim} & L_y^s & \longrightarrow & G/B \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} \\ \bullet & \longrightarrow & \{y\} & \longrightarrow & G/P \end{array}$$

The K-action induces an isomorphism $K \times^{K_y} L_y^s \xrightarrow{\sim} K_y \cdot L_y^s$. Thus,

$$D_K(K\cdot L_y^s) = D_K(K \times^{K_y} L_y^s) = D_{K_y}(L_y^s) = D_{K_y}(\mathbf{P}^1).$$[2]

As $|B\backslash G/K| < \infty$, the image of K_y in $\mathrm{Aut}(L_y^s)$ has dimension ≥ 1. Identify $\mathbf{P}^1$ with $\mathbf{A}^1 \sqcup \{\infty\}$. Modulo conjugation by an element of $\mathrm{Aut}(L_y^s) \simeq PGL_2$, there are four possibilities for the decomposition of $\mathbf{P}^1$ into K_y-orbits:

Case G: $\mathbf{P}^1$ (the action is transitive);
Case U: $\mathbf{P}^1 = \mathbf{A}^1 \sqcup \{\infty\}$;
Case T: $\mathbf{P}^1 = \{0\} \sqcup \mathbf{G}_m \sqcup \{\infty\}$;
Case N: $\mathbf{P}^1 = \{0,\infty\} \sqcup \mathbf{G}_m$; both $\{0\}$ and $\{\infty\}$ are fixed points of K_y^0.

Given an irreducible equivariant local system V_τ on a K-orbit $j\colon w \hookrightarrow G/B$, set

$$\mathcal{L}_\tau = j_{!*}V_\tau[d_\tau], \quad \text{where} \quad d_\tau = \dim(w).$$

Call $\mathcal{L}_\tau$ *clean* if $\mathcal{L}_\tau \simeq j_! V_\tau[d_\tau]$. Call $\mathcal{L}_\tau$ *cuspidal* if for each simple reflection s, each $v \neq w$ with $s \star v = w$, and each K-equivariant local system V_γ on v, the object $\mathcal{L}_\tau$ does *not* occur as a direct summand of ${}^pH^*(\pi^*\pi_*\mathcal{L}_\gamma)$, where π is as in (∗).[3]

Lemma 0.2.3. *Cuspidals are clean.*

[2] This is the analogue of the Lie theoretic principle that 'local phenomena is controlled by SL_2'.
[3] The term 'cuspidal' may be a bad choice: too many representation theoretic connotations.

Proof. Let $w \subseteq G/B$ be a K-orbit, and V_τ a local system on w such that $\mathcal{L}_\tau$ is cuspidal. It suffices to show that V_τ does not extend to a local system on any orbit of codimension 1 in the closure $\overline{w}$. Let v be such an orbit. By Corollary 0.2.1, there exists a simple reflection s such that $s \star v = w$. Let π be as in ($*$). By definition, $\mathcal{L}_\tau$ does not occur as a direct summand of ${}^pH^*(\pi^*\pi_*\mathcal{L}_\gamma)$ for any equivariant local system V_γ on v. Inspecting the cases **G**, **U**, **T** and **N** above, we see that V_τ must be a local system that does not extend to v (cf. [LV, Lemma 3.5 (e)]). □

0.3. **Mixed structures.** Given a variety X, write $M(X)$ for the category of mixed sheaves on X, and $DM(X)$ for the corresponding bounded derived category. In the characteristic 0 setting, 'mixed' should be understood in the context of M. Saito's theory of mixed Hodge modules [Sa]. If X is a variety over $\bar{\mathbf{F}}_q$, then mixed should be understood in the context of P. Deligne's Weil Conjectures machinery (see [BBD, §5.1.5]); in this setting we assume that X is the extension of scalars of a scheme X_0 over $\mathbf{F}_q$. If a linear algebraic group acts on X, write $M_G(X)$ (resp. $DM_G(X)$) for the corresponding category of mixed equivariant perverse sheaves (resp. mixed equivariant derived category). When dealing with mixed as well as ordinary categories, objects in mixed categories will be adorned with an ^{M}. Omission of the ^{M} will denote the classical object underlying the mixed structure.

We will call an object of $DM(\mathrm{pt})$ *Tate* if it is contained in the monoidal triangulated subcategory of $DM(\mathrm{pt})$ generated by the monoidal unit and $H^2(\mathbf{P}^1)$. An object $\mathcal{A}^M \in DM(X)$ will be called $*$*-pointwise Tate* if, for each point $i\colon \{x\} \hookrightarrow X$, $i^*\mathcal{A}^M$ is Tate. An object of $DM_G(X)$ will be called $*$-pointwise Tate if it is so under the forgetful functor $DM_G(X) \to DM(X)$.

Lemma 0.3.1. *Let π be as in ($*$). Then $\pi^*\pi_*$ preserves the class of $*$-pointwise Tate objects.*

Proof. Use the notation surrounding ($*$). Then the assertion reduces to the claim that if $\mathcal{A}^M \in M_{K_y}(L_y^s)$ is $*$-pointwise Tate, then $H^*(L_y^s; \mathcal{A}^M)$ is Tate. This is immediate from the possible K_y-orbit decompositions **G**, **U**, **T** and **N**. □

For each $\mathcal{L}_\tau$, there is a unique (up to isomorphism) mixed perverse sheaf $\mathcal{L}_\tau^M$ of weight d_τ which maps to $\mathcal{L}_\tau$ upon forgetting the mixed structure.

Proposition 0.3.2. *$\mathcal{L}_\tau^H$ is $*$-pointwise Tate.*

Proof. Work in G/B. The statement is true for cuspidals, since they are clean (Lemma 0.2.3). The general case follows by induction and Lemma 0.3.1. □

Comment: In the ℓ-adic case, we do not know if each ${}^pH^i(\pi^*\pi_*\mathcal{L}_\tau^M)$ is semisimple (in the mixed category). It is true (I haven't checked the details carefully though) for ${}^pH^1(\pi^*\pi_*\mathcal{L}_\tau^M)$ (if π is as in the induction), since each irreducible in ${}^pH^1(\pi^*\pi_*\mathcal{L}^\tau)$ occurs with multiplicity 1 (this can be double checked combinatorially using properties of Kazhdan-Lusztig-Vogan polynomials). Anyway, it would be nice to know this sort of semisimplicity (which undoubtedly holds) in general.

Proposition 0.3.3. *Let $i\colon v \hookrightarrow G$ be the inclusion of a $B \times K$-orbit. Then $i^*\mathcal{L}_\tau^H$ is pure.*

Proof. Work in G/K. According to [MS, §6.4], each B-orbit admits a contracting slice in the sense of [MS, §2.3.2]. This implies purity (see [MS, §2.3.2] or [KL, Lemma 4.5] or [So89, Proposition 1]). □

Given an algebraic group L acting on a variety X, set

$$\mathrm{Ext}^i_L(-,-) = \mathrm{Hom}_{D_L(X)}(-,-[i]).$$

The mixed sheaves $\mathcal{L}^H_\nu$ endow each $\mathrm{Ext}^\bullet_{B\times K}(\mathcal{L}_\tau, \mathcal{L}_\gamma)$ with a mixed structure.

Theorem 0.3.4. $\mathrm{Ext}^\bullet_{B\times K}(\mathcal{L}_\tau, \mathcal{L}_\gamma)$ *is Tate and pure of weight* $d_\gamma - d_\tau$.[4]

Proof. Work in G/K. Filtering $\mathrm{Ext}^\bullet_B(\mathcal{L}_\tau, \mathcal{L}_\gamma)$ by the orbit stratification, one sees that it suffices to argue that $\mathrm{Ext}^\bullet_B(i^*\mathcal{L}_\tau, i^!\mathcal{L}_\gamma))$ is pure and Tate for each B-orbit inclusion $i\colon u \hookrightarrow G/K$. Proposition 0.3.2 and Proposition 0.3.3 imply that both $i^*\mathcal{L}^H_\tau$ and $i^!\mathcal{L}^H_\gamma$ are pure and Tate. As H^*_L is pure and Tate, for any linear algebraic group L (cf. [D, §9.1]), the assertion follows. ☐

A comment/question: Consider the ℓ-adic setting. It would be nice to know if the Frobenius action on each $\mathrm{Ext}^i_{B\times K}(\mathcal{L}_\tau, \mathcal{L}_\gamma)$ is semisimple. I have no idea how to approach this.

References

[BBD] A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux Pervers*, Astérisque **100** (1982), 1-172.

[BL] J. Bernstein, V. Lunts, *Equivariant sheaves and functors*, Lecture Notes in Math. **1578**, Springer-Verlag, Berlin (1994).

[D] P. Deligne, *Théorie de Hodge III*, Pub I.H.É.S **44** (1974), 5-77.

[KL] G. Lusztig, D. Kazhdan, *Schubert varieties and Poincaré duality*, in Geometry of the Laplace operator, Proc. of Symposia in Pure Math XXXVI, AMS (1980).

[LV] G. Lusztig, D.A. Vogan, Jr., *Singularities of closures of K-orbits in flag manifolds*, Invent. Math. **71** (1983), 365-379.

[MS] J.G.M. Mars, T.A. Springer, *Hecke algebra representations related to spherical varieties*, Rep. Theory **2** (1998), 33-69.

[M] D. Miličić, *Localization and Representation Theory of Reductive Lie Groups*, book draft, available at http://www.math.utah.edu/~milicic/.

[RS] R.W. Richardson, T.A. Springer, *The Bruhat order on symmetric varieties*, Geom. Dedicata **35** (1990), 389-436.

[Sa] M. Saito, *Mixed Hodge modules*, Publ. RIMS **26** (1990), 221-333.

[So89] W. Soergel, $\mathfrak{n}$*-cohomology of simple highest weight modules on walls and purity*, Invent. Math. **98** (1989), 565-580.

[So] W. Soergel, *Langlands philosophy and Koszul duality*, in Proc. NATO ASI 2000 in Constanta, Algebra-Representation Theory, Kluwer (2001), 379-414.

[St] R. Steinberg, *Endomorphisms of linear algebraic groups*, Mem. A.M.S. **80** (1968).

[V] T. Vust, *Opération de groupes réductifs dans un type de cônes presque homogènes'*, Bull. Math. Soc. France **102** (1974), 317-334.

[4] **Warning:** the non-equivariant analogue of this result is false!