SUMMANDS OF BOTT-SAMELSON MOTIVES

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1. For my own sake, let me state the question you asked. The notation is as follows: G is connected reductive, $B \subset G$ a Borel, W the Weyl group, and $\ell\colon W \to \mathbf{Z}_{\geq 0}$ the length function. It's assumed that a maximal torus in B has been fixed, so we have simple reflections and the like. For a simple reflection s, I will write $P_s \supset B$ for the associated minimal parabolic.

Let $w \in W$, and fix a reduced word

$$w = s_0 \cdots s_k.$$

To this reduced word is associated the so-called Bott-Samelson resolution

$$\pi\colon P_{s_0} \times^B P_{s_1} \times^B \cdots \times^B P_{s_k}/B \to G/B.$$

By the Decomposition Theorem

$$\pi_* \underline{\mathbf{Q}} \approx IC_w \bigoplus_{x \in J} IC_x,$$

where $\approx$ means modulo shifts, and IC_x means the intersection complex on the Schubert variety associated to x. The question is to

$$\text{determine } J.$$

2. Ok, I am now going to rephrase the question in the Hecke algebra. To prevent confusion, let me define the Hecke algebra, etc. - conventions in the literature differ a lot, this can cause some re-scaling by q, $q^{\frac{1}{2}}$ and things of that nature. I would suggest skipping defining junk section and jumping to the section titled 'rephrasing the question', referring back as needed.

The Hecke algebra will be the $\mathbf{Z}[v, v^{-1}]$ algebra generated by T_w, $w \in W$, and relations

$$T_x T_y = T_{xy} \qquad \text{if } \ell(xy) = \ell(x) + \ell(y),$$
$$(T_s + 1)(T_s - v^{-2}) = 0 \qquad \text{for each simple reflection } s.$$

The v^2 corresponds to the inverse of the Tate twist (-1) - let's not worry about the meaning of 'half Tate twist'. The T_x correspond to !-pushforwards of the constant sheaf of the Schubert cell corresponding to x (so no shift here, i.e., these aren't pervers). The quadratic relation basically just encodes the cohomology of $\mathbf{P}^1$.

In order to work with elements in this algebra that correspond to perverse sheaves, and to ensure that the (Kazhdan-Lusztig) basis elements that correspond to intersection complexes being of weight 0, it is convenient to work with the following elements.

Set

$$H_x = v^{\ell(x)} T_x.$$

Now we have the bar involution (corresponding to Verdier duality), defined by

$$\bar{v} = v^{-1}, \qquad \bar{H}_x = H_{x^{-1}}^{-1}.$$

Now the standard result is that for each $x \in W$, there exists a self-dual (with respect to the bar involution) element C_x such that

$$C_x \in H_x + \sum_{y<x} v\mathbf{Z}[v]H_y.$$

The C_x are basically the Kazhdan-Lusztig basis (i.e., correspond to intersection complexes of weight 0).

3. **The question rephrased.** The original Bott-Samelson question now becomes the following

$$C_{s_0} \cdots C_{s_k} = C_w + \sum_{x<w} h_x C_x.$$

Determine when $h_x \neq 0$. Ok, as $C_{s_1} \cdots C_{s_k}$ is going to be a similar sum of C_ys, we could do this inductively by figuring out what $C_s C_x$ is, for a simple reflection s and x arbitrary. An 'explicit' formula for this is easy to derive

$$C_s C_x = \begin{cases} C_{sx} + \sum_{\substack{y<x,\\ sy<y}} \mu(y,x) C_y & \text{if } sx > x, \\ (v+v^{-1})C_x & \text{if } sx < x. \end{cases}$$

Here $\mu(y,x)$ is defined as follows. Define polynomials $h_{y,x} \in \mathbf{Z}[v]$ by

$$C_x = \sum_y h_{y,x} H_y.$$

Now

$$\mu(y,x) \text{ is the coefficient of } v \text{ in } h_{y,x}.$$

Essentially (up to scaling by a power of v), the $h_{y,x}$ are Kazhdan-Lusztig polynomials, and $\mu(y,x)$ is their 'leading coefficient'. I have chosen the notation so that my $\mu(y,x)$ coincides with Kazhdan-Lusztig's μ (even though I am using different conventions for the Hecke algebra, etc.). This μ function is pretty infamous, and will be the same in the literature on Kazhdan-Lusztig polynomials, regardless of the conventions being used. I.e., if you see a μ function in a paper on KL-polynomials, it is highly likely that it is the same μ as here, regardless of conventions, definitions, etc.

So an inductive version of the question becomes

$$\text{determine when } \mu(y,x) \neq 0?$$

As far as I know, even though a lot of effort has been expended on this question, not much is known. There seem to be some simple *necessary* conditions for $\mu(y,x) \neq 0$, but these are known to not be sufficient. I am pretty sure that, apart from some limited special cases, no sufficient conditons are known. In fact, given that this μ function also plays a role in the inductive construction of the KL-basis, any nice combinatorial rule describing it, or indicating when it is non-zero, would probably yield combinatorial insight into KL-polynomials.

Essentially all I am saying is that this seems to be a hard problem, and at least for the moment I give up. All I am going to now do is mention one easy condition that guarantees the vanishing of $\mu(y,x)$, and quickly explain what μ has to do with *cells* (another object that is hard to get a combinatorial handle on).

The condition is simple:

$$\text{if } \ell(x) - \ell(y) \text{ is even, then } \mu(y, x) = 0.$$

As far as the relation with cells goes. Consider the following problem: instead of asking if IC_x occurs as a summand using the Bott-Samelson resolution for a fixed reduced word for w, one can ask if IC_x occurs as a summand using a Bott-Samelson resolution corresponding to some reduced word for w (so same question as the original, except don't fix the reduced word). Translating into the Hecke algebra, and into the above framework, this leads to the following consideration.

Write $x \leq_L y$ if there exist a simple reflection s such that C_y occurs with non-zero coefficient in the expansion of $C_s C_x$ in the KL-basis (the C_w). Extend $\leq_L$ to a partial order on W. The equivalence classes corresponding to this order are precisely what are called the left cells of W.

The Appalachians