COMPUTING EXTENSIONS

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0.1. **Notational comments.** I will be deliberately vague about what a 'space' is and what a 'sheaf' on a space is. Everything that follows is formulated functorially (dare we say motivically) enough that it goes through in the following rough generality: some category fibred over the category of 'reasonable spaces' with fibre a triangulated category (satisfying suitable descent conditions), objects of the latter being called 'sheaves'. I also require a limited functor formalism for the triangulated category of sheaves: monoidal structure, pullback and pushforward along inclusions, extension by zero, restriction with supports, along with the standard adjunctions and distinguished triangles between these. The constant sheaf on a space X, denoted $\underline{X}$, will always mean the pullback of the unit object over a point. For the sake of intuition, the reader should take 'space' = 'complex variety', and 'sheaf' = 'constructible complex of sheaves in the classical topology'.

0.2. **Dévissage step.** Let X be a space, $j\colon U \hookrightarrow X$ the inclusion of an open subspace, and $i\colon Y \hookrightarrow X$ the inclusion of the closed complement $Y = X - U$. Let L, N be sheaves on X. Then we have the familiar long exact sequences:

$$\cdots \to \mathbb{H}^\bullet(j_!j^*L) \to \mathbb{H}^\bullet(L) \to \mathbb{H}^\bullet(i_*i^*L) \to \cdots \tag{0.2.1}$$

$$\cdots \to \mathbb{H}^\bullet(i_*i^!L) \to \mathbb{H}^\bullet(L) \to \mathbb{H}^\bullet(j_*j^*L) \to \cdots \tag{0.2.2}$$

$$\cdots \to \mathrm{Ext}^\bullet(i^*L, i^!N) \to \mathrm{Ext}^\bullet(L,N) \to \mathrm{Ext}^\bullet(j^*L, j^*N) \to \cdots \tag{0.2.3}$$

Consider the following conditions:

($*$) the sequence (0.2.1) splits into short exact sequences:

$$0 \to \mathbb{H}^\bullet(j_!j^*L) \to \mathbb{H}^\bullet(L) \to \mathbb{H}^\bullet(i_*i^*L) \to 0$$

(!) the sequence (0.2.2) splits into short exact sequences:

$$0 \to \mathbb{H}^\bullet(i_*i^!L) \to \mathbb{H}^\bullet(L) \to \mathbb{H}^\bullet(j_*j^*L) \to 0$$

($*-!$) the sequence (0.2.3) splits into short exact sequences:

$$0 \to \mathrm{Ext}^\bullet(i^*L, i^!N) \to \mathrm{Ext}^\bullet(L,N) \to \mathrm{Ext}^\bullet(j^*L, j^*N) \to 0$$

(F) the map $\mathrm{Ext}^\bullet(i^*L, i^!N) \to \mathrm{Hom}(\mathbb{H}^\bullet(i^*L), \mathbb{H}^\bullet(i^!N))$ is injective, and the map $\mathrm{Ext}^\bullet(j^*L, j^*N) \to \mathrm{Hom}(\mathbb{H}^\bullet(j^*L), \mathbb{H}^\bullet(j^*N))$ is an isomorphism.

Throughout, hypercohomology $\mathbb{H}^\bullet(L) = \mathrm{Ext}^\bullet(\underline{X}, L)$ is considered as a functor to graded $H^*(X) = \mathrm{Ext}^\bullet(\underline{X}, \underline{X})$-modules.

Lemma 0.3. *Let $L \in D^b(X)$ and $M \in D^b(U)$. Suppose L satisfies $(!)$, and*

$$\mathrm{Ext}^\bullet(j^*L, M) \to \mathrm{Hom}(\mathbb{H}^\bullet(j^*L), \mathbb{H}^\bullet(M))$$

is an injection. Then the map

$$\mathrm{Ext}^\bullet(L, j_*M) \to \mathrm{Hom}(\mathbb{H}^\bullet(L), \mathbb{H}^\bullet(j_*M))$$

is injective.

Proof. Applying $\mathrm{Hom}(-, \mathbb{H}^\bullet(j_*M))$ to (!) we obtain an injection

$$\mathrm{Hom}(\mathbb{H}^\bullet(j^*L), \mathbb{H}^\bullet(M)) \hookrightarrow \mathrm{Hom}(\mathbb{H}^\bullet(L), \mathbb{H}^\bullet(j_*M)).$$

I leave it to the reader to verify that the composition

$$\mathrm{Ext}^\bullet(L, j_*M) \xrightarrow{\sim} \mathrm{Ext}^\bullet(j^*L, M) \hookrightarrow \mathrm{Hom}(\mathbb{H}^\bullet(j^*L), \mathbb{H}^\bullet(M)) \hookrightarrow \mathrm{Hom}(\mathbb{H}^\bullet(L), \mathbb{H}^\bullet(j_*M))$$

is the map in question (use the standard properties of adjunction maps). □

Proposition 0.4. *Suppose $(*-!)$ and (F) hold. Further, assume L satisfies $(*)$, and both L, N satisfy $(!)$. Then the map*

$$\mathrm{Ext}^\bullet(L, N) \to \mathrm{Hom}(\mathbb{H}^\bullet(L), \mathbb{H}^\bullet(N))$$

is injective.

Proof. We have a commutative diagram:

$$\begin{array}{ccc}
0 & & \\
\downarrow & & \\
\mathrm{Ext}^\bullet(i^*L, i^!N) & \longrightarrow & \mathrm{Hom}(\mathbb{H}^\bullet(i^*L), \mathbb{H}^\bullet(i^!N)) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^\bullet(L, N) & \longrightarrow & \mathrm{Hom}(\mathbb{H}^\bullet(L), \mathbb{H}^\bullet(N)) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^\bullet(L, j_*j^*N) & \longrightarrow & \mathrm{Hom}(\mathbb{H}^\bullet(L), \mathbb{H}^\bullet(j_*j^*N)) \\
\downarrow & & \\
0 & &
\end{array}$$

The left vertical column is exact by $(*-!)$. The top horizontal map is injective by (F). Now take $M = j^*N$. By (!) for L and (F), we are in the situation of the previous Lemma. So the bottom horizontal map is also injective. To complete the proof it suffices to show that the top vertical map in the second column is injective. Let $\phi\colon \mathbb{H}^\bullet(i^*L) \to \mathbb{H}^\bullet(i^!N)$ be non-zero. Then the map in question sends ϕ to the composition

$$\mathbb{H}^\bullet(L) \twoheadrightarrow \mathbb{H}^\bullet(i_*i^*L) \xrightarrow{\phi} \mathbb{H}^\bullet(i_*i^!N) \hookrightarrow \mathbb{H}^\bullet(N).$$

The first map in this composition is surjective by $(*)$ for L, and the last map is injective by (!) for N. □

Remark 0.5. The basic idea of the above argument is essentially stolen from [BGS, Proposition 3.4.2]. However, the spectral sequence used in *loc. cit.* makes it hard to keep track of the various maps involved.

Remark 0.6. It is clear that for the purposes of the Proposition we could weaken the isomorphism in (F) to an injection, but I do not know of any situation where such generality is helpful.

With a bit more geometric input we can strengthen the injection above to an isomorphism. Replace (F) by

> (FF) Both of the evident maps $\mathrm{Ext}^\bullet(i^*M, i^!N) \to \mathrm{Hom}(\mathbb{H}^\bullet(i^*M), \mathbb{H}^\bullet(i^!N))$ and $\mathrm{Ext}^\bullet(j^*L, j^*N) \to \mathrm{Hom}(\mathbb{H}^\bullet(j^*L), \mathbb{H}^\bullet(j^*N))$ are isomorphisms.

We also need transverse slices:

> (S) there exists a closed embedding $s\colon S \hookrightarrow X$ which is obtained locally on X by embedding X into a smooth space Z and then S is the intersecton $S' \cap X$, where $S' \subseteq Z$ is a closed subspace. Further, S has empty intersection with Y and intersects U transversally in a contractible subspace.

Proposition 0.7. *Suppose L, N are smooth along the stratification $X \supset Y \supset \varnothing$. Further, assume*

(i) *L satisfies $(*)$ and $(!)$;*
(ii) *$(*-!)$, (FF) and (S) hold;*
(iii) *N satisfies $(!)$.*

Then the map

$$\mathrm{Ext}^\bullet(L, N) \to \mathrm{Hom}(\mathbb{H}^\bullet(L), \mathbb{H}^\bullet(N))$$

is an isomorphism.

Proof. The transverse slice of (S) determines a cohomology class $[S] \in H^*(X)$. For any sheaf M that is smooth along the stratification $X \supset Y \supset \varnothing$, the action of $[S]$ on $\mathbb{H}^\bullet(M)$ (modulo grading shifts) is given by:

$$\mathbb{H}^\bullet(M) \to \mathbb{H}^\bullet(s_*s^*M) \xrightarrow{\sim} \mathbb{H}^\bullet(s_*s^!M) \to \mathbb{H}^\bullet(M),$$

where the middle isomorphism is a canonical isomorphism obtained from the transversality assumption (or can be taken as the abstract/formal definition of transversality). If a sheaf M satisfies (!), then the action of $[S]$ on $\mathbb{H}^\bullet(M)$ kills $\mathbb{H}^\bullet(i_*i^!M)$, since $S \cap Y = \varnothing$. In fact, as $S \cap U$ is non-empty and contractible, we infer $\mathbb{H}^\bullet(i_*i^!M)$ is precisely the kernel of the action of $[S]$. Similar reasoning shows that if M satisfies $(*)$, then the image of the action of $[S]$ is $\mathbb{H}^\bullet(j_!j^*M)$.

In our situation this implies that a morphism $\phi\colon \mathbb{H}^\bullet(L) \to \mathbb{H}^\bullet(N)$ must map $\mathbb{H}^\bullet(i_*i^!L)$ to $\mathbb{H}^\bullet(i_*i^!N)$. Thus, ϕ induces a morphism

$$\pi(\phi)\colon \mathbb{H}^\bullet(j_*j^*L) \simeq \mathbb{H}^\bullet(L)/\mathbb{H}^\bullet(i_*i^!L) \to \mathbb{H}^\bullet(N)/\mathbb{H}^\bullet(i_*i^!N) \simeq \mathbb{H}^\bullet(j_*j^*N).$$

Consequently, we obtain a commutative diagram:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{Ext}^\bullet(i^*L, i^!N) & \xrightarrow{\sim} & \mathrm{Hom}(\mathbb{H}^\bullet(i^*L), \mathbb{H}^\bullet(i^!N)) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^\bullet(L, N) & \longrightarrow & \mathrm{Hom}(\mathbb{H}^\bullet(L), \mathbb{H}^\bullet(N)) \\
\downarrow & & \downarrow \pi \\
\mathrm{Ext}^\bullet(j^*L, j^*N) & \xrightarrow{\sim} & \mathrm{Hom}(\mathbb{H}^\bullet(j^*L), \mathbb{H}^\bullet(j^*N)) \\
\downarrow & & \\
0 & &
\end{array}$$

The top and bottom horizontal maps are isomorphisms by (FF). The left column is exact by $(*-!)$. The top half of the right column is exact by $(*)$ for L and (!) for N (as in the proof of the previous Proposition). To complete the proof it suffices to show that the right column is exact in the middle. Let $\phi\colon \mathbb{H}^\bullet(L) \to \mathbb{H}^\bullet(N)$ be such that $\pi(\phi) = 0$. Then, using that L satisfies $(*)$, ϕ must map $\mathbb{H}^\bullet(j_!j^*L)$ to 0, and must map $\mathbb{H}^\bullet(L)/\mathbb{H}^\bullet(j_!j^*L) \simeq \mathbb{H}^\bullet(i_*i^*L)$ to $\mathbb{H}^\bullet(i_*i^!N) \subseteq \mathbb{H}^\bullet(N)$. Thus, ϕ induces a morphism $\mathbb{H}^\bullet(i^*L) \to \mathbb{H}^\bullet(i^!N)$. This latter morphism is mapped to ϕ in the right column of our commutative diagram. □

Remark 0.8. The idea to use transverse slices comes from [Gi].

References

[BGS] A. Beilinson, V. Ginzburg, W. Soergel, *Koszul duality patterns in Representation Theory*, Journal of A.M.S. **9** (1996), 473-527.

[Gi] V. Ginzburg, *Perverse sheaves and* $\mathbf{C}^*$*-actions*, Journal of A.M.S **4**, No. 3 (1991), 483-490.

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