# HODGE CONJECTURE SILLINESS

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Let $X$ be a variety. Write $CH_*(X)_{\mathbf{Q}}$ for its rational Chow groups, and $H_*(X;\mathbf{Q})$ for its Borel-Moore homology. If a group (always assumed to be algebraic) acts on $X$, write $CH_*^G(X)_{\mathbf{Q}}$ for its equivariant Chow groups, and $H_*^G(X;\mathbf{Q})$ for its equivariant Borel-Moore homology. Write $A_G^*$ for the rational equivariant Chow cohomology of a point, and $H_G^*$ for the equivariant cohomology ring of a point. If $G$ is linear algebraic, then the cycle map is a degree doubling isomorphism $A_G^* \xrightarrow{\sim} H_G^*$. Consequently, the following is manifest.

**Lemma 1.** *Let $G$ be a linear algebraic group, and $K \subset G$ a closed subgroup. Then the cycle class map*

$$CH_i^G(G/K)_{\mathbf{Q}} \twoheadrightarrow W_{-2i}H_{2i}^G(G/K;\mathbf{Q})$$

*is a surjection onto the lowest weight part of equivariant homology.*

**Theorem 2.** *Let $G$ be a linear algebraic group acting on a variety $X$ with finitely many orbits. Then the cycle class map*

$$CH_i^G(X)_{\mathbf{Q}} \twoheadrightarrow W_{-2i}H_{2i}^G(X;\mathbf{Q})$$

*is a surjection onto the lowest weight part of equivariant homology.*

*Proof.* Proceed by induction on the number of orbits. The case of a single orbit is the previous Lemma. In general, let $U \subset X$ be an open orbit. Let $Z = X - U$ be the closed complement. Then we have a morphism of exact sequences

$$\begin{array}{ccccccc}
CH_i^G(Z)_{\mathbf{Q}} & \longrightarrow & CH_i^G(X)_{\mathbf{Q}} & \longrightarrow & CH_i^G(U)_{\mathbf{Q}} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
W_{-2i}H_{2i}^G(Z;\mathbf{Q}) & \longrightarrow & W_{-2i}H_{2i}^G(X;\mathbf{Q}) & \longrightarrow & W_{-2i}H_{2i}^G(U;\mathbf{Q}) & \longrightarrow & 0
\end{array}$$

The outer vertical arrows are surjective, hence so must be the middle one. □

The non-equivariant analogue of this Theorem is also true. But to keep things simple and frivolous, let's content ourselves with the following.

**Theorem 3.** *Let $G$ be a linear algebraic groups acting on a variety $X$ with finitely many orbits. If $X$ is smooth and projective, then the cycle class map*

$$CH_i(X)_{\mathbf{Q}} \twoheadrightarrow H_{2i}(X;\mathbf{Q})$$

*is a surjection. In particular, the Hodge conjecture holds for $X$.*

*Proof.* The usual formality formalism yields that the natural maps

$$\mathbf{Q} \otimes_{A_G^*} CH_*^G(X)_{\mathbf{Q}} \xrightarrow{\sim} CH_*(X)_{\mathbf{Q}}, \qquad \mathbf{Q} \otimes_{H_G^*} H_*^G(X;\mathbf{Q}) \xrightarrow{\sim} H_*(X;\mathbf{Q})$$

are isomorphisms. So the claim is immediate from the previous result. □

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