SOME FORMAL ARGUMENTS

1. Smooth base change

Assume that we have functors f^*, f_* along with the adjunctions between them in the equivariant setting. Further, assume that the forgetful functor $\mathrm{For}\colon DM_G(X) \to DM(X)$ is a functor of triangulated categories and commutes with these functors, adjunctions and exchange properties.

The claim is that if For is conservative (i.e., $\mathrm{For}(\mathcal{A}) = 0$ if and only if $\mathcal{A} = 0$), then some things like smooth base change are formal consequences of their non-equivariant analogues. I now hope that Matthias immediately says that it is clear that For is conservative! (This should be a special case of a more general statement along the lines of: f^* for surjective f is conservative?)

So assume that For is conservative, and argue as follows.

Lemma 0.1. *A morphism $\phi\colon \mathcal{A} \to \mathcal{B}$ in $DM_G(X)$ is an isomorphism if and only if $\mathrm{For}(\phi)$ is an isomorphism.*

Proof. Necessity is clear. Let's demonstrate sufficiency. So assume $\mathrm{For}(\phi)$ is an isomorphism. Then $\mathrm{For}(\mathrm{cone}(\phi)) = \mathrm{cone}(\mathrm{For}(\phi)) = 0$. As For is conservative, this implies $\mathrm{cone}(\phi) = 0$. So ϕ is an isomorphism. □

This immediately yields the smooth base change isomorphism - given a Cartesian square

$$\begin{array}{ccc} Z & \xrightarrow{\tilde{g}} & X \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & S \end{array}$$

with g smooth, the base change morphism is the composition

$$g^* f_* \to \tilde{f}_* \tilde{f}^* g^* f_* = \tilde{f}_* \tilde{g}^* f^* f_* \to \tilde{f}_* \tilde{g}^*$$

with the arrows given by adjunctions and the equality given by exchange properties. We know that applying For to this yields an isomorphism. So by the Lemma we are done.

The point is that as soon as we know that a map in the non-equivariant setting 'lifts' to something equivariant, then we are done as far as proving isomorphisms is concerned.

For general motives on diagrams of schemes, I think the point should be that one can check for isomorphism of objects by testing via pulling back along (any?) surjective maps.

2. Induction equivalence

This section doesn't utilize the discussion in the previous one. The goal is to derive the induction equivalence in a formal way from the quotient equivalence.

Proposition 0.2 (Induction equivalence)**.** *Let G be a linear algebraic group, let $\phi\colon H \hookrightarrow G$ be a closed subgroup. For an H-scheme X, let $f\colon X \to G \times_H X$, $x \mapsto [(e,x)]$ be the natural inclusion. Then*

$$(\phi, f)^*\colon DM_G(G \times^H X) \xrightarrow{\sim} DM_H(X),$$
$$(\phi, f)_*\colon DM_H(X) \xrightarrow{\sim} DM_G(G \times^H X),$$

are mutually inverse equivalences.

Proof. Let $\pi_G\colon G \times H \to G$ and $\pi_H\colon G \times H \to H$ be the evident projections. Consider $G \times H$ acting on $G \times X$ via $(g,h)\cdot(g',x) = (gg'h^{-1}, hx)$. Let $p\colon G \times X \to X$ be the projection, and $q\colon G \times X \to G \times^H X$ the quotient map. By the quotient equivalence the composition

$$DM_G(G \times^H X) \xrightarrow{(\pi_G,q)^*} DM_{G\times H}(G \times X) \xrightarrow{(\pi_H,p)_*} DM_H(X)$$

is an equivalence, with inverse

$$DM_H(X) \xrightarrow{(\pi_H,p)^*} DM_{G\times H}(G \times X) \xrightarrow{(\pi_G,q)_*} DM_G(G \times^H X).$$

Consequently, it suffices to show

$$(\pi_H, p)^* \circ (\phi, f)^* = (\pi_G, q)^*.$$

Let

$$\tilde{f}\colon X \to G \times X, \qquad x \mapsto (e,x),$$

and

$$\delta_H\colon H \to G \times H, \qquad h \mapsto (h,h).$$

Then

$$(\delta_H, \tilde{f})^* \circ (\pi_H, p)^* = \mathrm{id}.$$

As $(\pi_G, p)^*$ is an equivalence, this implies

$$(\pi_H, p)^* \circ (\delta_H, \tilde{f})^* = \mathrm{id}.$$

Consequently,

$$\begin{aligned}(\pi_H, p)^* \circ (\phi, f)^* &= (\pi_H, p)^* \circ (\delta_H, \tilde{f})^* \circ (\pi_G, q)^* \\ &= (\pi_G, q)^*.\end{aligned}$$

□