THE BASIC OBSERVATION: ALTERNATE PROOF

R. VIRK

1. **The Basic Observation.** The following Lemma is well known.

Lemma 1.1. *Let G be a linear algebraic group. Then $H^*(G)$ is Tate.*

Proof. We may assume G is connected reductive (Levi decomposition). □

Basic Observation. *Let G be a linear algebraic group acting on a variety X. If $H^*(X)$ is Tate, then the G-equivariant cohomology $H^*_G(X)$ is Tate.*

Proof. Argue by contradiction. We may assume G is a torus. Let n be minimal with the property that $H^n_G(X)$ contains a non-Tate class. Let EG be an N-acyclic approximation, $N \gg 0$, to the universal bundle on the classifying space BG (yep, I am abusing notation here). Consider the Leray spectral sequence associated to the G-torsor

$$\pi\colon EG \times X \to EG \times^G X.$$

As G is a torus, this torsor is *Zariski* locally trivial. In particular, the $VMHS$ $R^q\pi_*\mathbf{Q}$ has Tate fibre. Now the $E_2^{n,0}$-term is

$$E_2^{n,0} = H^n(EG \times^G X) = H^n_G(X).$$

By the minimality of n, any non-Tate class in $E_2^{n,0}$ must survive to the abutment (for $i \geq 2$, the entry $E_i^{n,0}$ is always in the kernel of the differential, but there are no non-Tate classes above and strictly to the left of this entry). But the latter is $H^n(EG \times X) = H^n(X)$. □

Now that I think about it, it is probably simpler to look at the fibration

$$X \hookrightarrow EG \times^G X \twoheadrightarrow BG.$$

Or in the G/K case, just look at the fibration

$$K \hookrightarrow G \twoheadrightarrow G/K.$$

Take K to be a torus to make it Zariski locally trivial (so no funny business with strange Hodge structures on the fibres).

The Appalachians