NOTES ON MOTIVIC MODELS FOR CATEGORY $\mathcal{O}$

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Contents

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These are notes on portions of [SW].

0.1. **Conventions.** A *variety* will mean a separated scheme of finite type over some fixed field k. In order to emphasize the base field I will often write X/k for a variety X over k. As just done, I will often abuse notation and write 'k' where strictly speaking one should write 'Spec(k)'.

A *stratification* of a variety X will mean a partition of X into disjoint locally closed smooth subvarieties (strata) such that the closure of a stratum is the union of strata.

We write $DM(X)$ for the category of constructible Beilinson motives (rational coefficients) on X (see [CD]). Note: our $DM(X)$ is denoted $DM_{\mathrm{B,c}}(X)$ in [CD]. This is a monoidal triangulated category, and the assignment $X \mapsto DM(X)$ admits the standard yoga of the 'six-functors'. The unit object in $DM(X)$ (the "constant sheaf") will be denoted $\underline{X}$. The n-th Tate twist will be denoted (n).

0.2. **Beilinson-Soulé vanishing.** A variety X/k will be said to satisfy *Beilinson-Soulé vanishing* if $\mathrm{Hom}(\underline{X}, \underline{X}[i](j)) = 0$ for all $i < 0$ and $j > 0$. The connection with the Beilinson-Soulé conjectures in algebraic K-theory is the following. If S is smooth, then (according to [CD, Corollary 14.2.14]):

$$\mathrm{Hom}(\underline{S}, \underline{S}[i](j)) = K_{2j-i}(S)^{(j)}_{\mathbf{Q}}.$$

Here $K_*(-)^{(j)}_{\mathbf{Q}}$ denotes the j-th Adams graded part of rational (algebraic) K-theory.

Let $\mathrm{Vec}_{\mathbf{Q}}$ and $\mathrm{Vec}_{\mathbf{Q}_\ell}$ denote the categories of finite-dimensional $\mathbf{Q}$- and $\mathbf{Q}_\ell$-vector spaces respectively. For ℓ prime to the characteristic of k one has the ℓ-adic realization functors $r_\ell \colon DM(k) \to D^b(\mathrm{Vec}_{\mathbf{Q}_\ell})$. For k of characteristic 0, each embedding $\iota \colon k \to \mathbf{C}$ yields a Betti realization functor $r_\iota \colon DM(k) \to D^b(\mathrm{Vec}_{\mathbf{Q}})$.

These are all tensor triangulated functors. There are canonical identifications $r_\iota \otimes \mathbf{Q}_\ell \xrightarrow{\sim} r_\ell$. We will write r for any of these realization functors.

Let $DMT(X)^{\leq 0}$ (resp. $DMT(X)^{\geq 0}$) denote the subcategory of $DMT(X)$ consisting of objects that admit a filtration by the $\underline{X}[i](j)$, $j \in \mathbf{Z}$, $i \in \mathbf{Z}_{\geq 0}$ (resp. $i \in \mathbf{Z}_{\leq 0}$). If k satisfies Beilinson-Soulé vanishing, then this yields a t-structure [L] on $DMT(X)$ with heart

$$MT(X) := DMT(X)^{\leq 0} \cap DMT(X)^{\geq 0}.$$

The following is clear.

Proposition 0.2.1. *Assume k satisfies Beilinson-Soulé vanishing. Then the functor r restricted to $MT(k)$ is faithful.*

Proof. An object in $MT(k)$ admits a finite filtration with subquotients of the form $\underline{\mathrm{Spec}(k)}(j)$. The assertion follows by induction on the length of this filtration. □

Remark. See [B] for a beautiful proof that applies more generally to the (as yet conjectural) full motivic t-structure on $DM(k)$.

Warning. r is *not* faithful on $DMT(k)$.

Corollary 0.2.2. *If X/k is of Tate type and k satisfies Beilinson-Soulé vanishing, then X satisfies Beilinson-Soulé vanishing.*

Proof. Let $a\colon X \to \mathrm{Spec}(k)$ be the structure map. Then

$$\mathrm{Hom}(\underline{X}, \underline{X}[i](j)) = \mathrm{Hom}(\underline{\mathrm{Spec}(k)}, a_*\underline{X}[i](j)).$$

As X/k is of Tate type and k satisfies Beilinson-Soulé vanishing, it suffices to argue that $a_*\underline{X} \in DMT(k)^{\geq 0}$. This is immediately seen via a realization r. □

Call a variety $a\colon X \to \mathrm{Spec}(k)$ of *Tate type* if $a_*\underline{X} \in DMT(k)$. A large class of varieties of Tate type is provided by *linear varieties*. Linear varieties (over k) are defined inductively as follows:

(i) affine n-space $\mathbf{A}^n$ is a linear variety;
(ii) the complement of a linear variety embedded (as an open or closed subvariety) in another linear variety is a linear variety;
(iii) a variety stratified by linear varieties is a linear variety.

Every variety that admits a solvable group action with finitely many orbits is a linear variety, since each orbit is isomorphic to some $\mathbf{G}_m^a \times \mathbf{A}^b$. Other examples include the intersections of Schubert varieties with opposite Schubert varieities.

0.3. **Linear stratifications.** Let $X = \bigsqcup_{w \in W} X_w$ be a stratified variety. For each $w \in W$, let $i_w\colon X_w \hookrightarrow X$ be the inclusion. Let

$$\Delta_w := i_{w!}\underline{X_w}[\dim X_w],$$
$$\nabla_w := i_{w*}\underline{X_w}[\dim X_w].$$

Let $DMT_\Delta(X)$ (resp. $DMT_\nabla(X)$) be the full triangulated subcategory of $DM(X)$ generated by Tate twists of Δ_w (resp. ∇_w), $w \in W$. Call the stratification *Whitney-linear* if the following conditions are satisfied:

(i) the categories $DMT_\Delta(X)$ and $DMT_\nabla(X)$ coincide;
(ii) each stratum X_w is a linear variety;

(iii) each inclusion $i_w\colon X_w \hookrightarrow X$ is affine.

In this situation we set $DM_\Diamond(X) := DM_\Delta(X) = DM_\nabla(X)$.

Assume X is Whitney-linear stratified and that the base field k satisfies Beilinson-Soulé vanishing. Set

$$DM_\Diamond(X)^{\leq 0} := \{M \in DM_\Diamond(X) \mid i_w^* M \in DMT(X_w)^{\leq -\dim(X_w)} \text{ for all } w \in W\},$$
$$DM_\Diamond(X)^{\geq 0} := \{M \in DM_\Diamond(X) \mid i_w^! M \in DMT(X_w)^{\geq -\dim(X_w)} \text{ for all } w \in W\}.$$

Then the the gluing formalism for t-structures yields that this is a t-structure on $DMT_\Diamond(X)$. Let

$$PMT_\Diamond(X) := DMT_\Diamond(X)^{\leq 0} \cap DMT_\Diamond(X)^{\geq 0}.$$

0.4. **The case of a finite field and gradings.** The realization functors on $DM(k)$ extend to the categories $DM(X)$. In particular, if k is a finite field, then we have a realization functor $DM(X) \to D^b(X \otimes_k \bar{k}; \mathbf{Q}_\ell)$, where the right hand side is the bounded derived category of constructible complexes of ℓ-adic (ℓ different from $\operatorname{char}(k)$) sheaves. Realization commutes with the six-functor formalism.

For the remainder of this section our base field will be a finite field $\mathbf{F}_q$ with q elements. The K-theory of finite fields is particularly simple:

$$K_i(\mathbf{F}_q) = \begin{cases} \mathbf{Z} & \text{for } i = 0; \\ \mathbf{F}_q^\times & \text{for } i = 1; \\ 0 & \text{otherwise.} \end{cases}$$

In particular $\mathbf{F}_q$ satisfies Beilinson-Soulé vanishing. Moreover, $MT(\mathbf{F}_q)$ is equivalent to the category of finite dimensional graded vector spaces.

We need a tiny bit of additional formalism. A *grading* on a triangulated category $\mathcal{T}_0$ is the data of a triangulated category $\mathcal{T}$ equipped with an auto-equivalence $(1)\colon \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ along with a functor $\omega\colon \mathcal{T} \to \mathcal{T}_0$ such that:

(i) there is a canonical isomorphism $\omega \circ (1) \simeq \omega$;

(ii) the map

$$\bigoplus_i \operatorname{Hom}_{\mathcal{T}}(M, N(i)) \to \operatorname{Hom}_{\mathcal{T}_0}(\omega(M), \omega(N))$$

is an isomorphism for all $M, N \in \mathcal{T}$. Here, (i) is (1) iterated i-times.

Theorem 0.4.1. *Let X be a Whitney-linear stratified variety over $\mathbf{F}_q$. Then realization $r_\ell\colon DMT_\Diamond(X) \to D^b(X \otimes_{\mathbf{F}_q} \bar{\mathbf{F}}_q; \mathbf{Q}_\ell)$ is a grading.*

Proof. Induction reduces the assertion to the case consisting of a single stratum. As all varieties involved are linear, we are further reduced to the case of $\mathbf{A}^n$. By $\mathbf{A}^n$-homotopy invariance this reduces to the case $X = \operatorname{Spec}(\mathbf{F}_q)$ where the assertion is evident. □

0.5. **The case of Schubert stratification.** Let G be a reductive linear algebraic group. Let $X = G/B$ be its flag variety stratified by Schubert cells X_w, $w \in W$, where W is the Weyl group. In order to apply the formalism of the previous sections to the Schubert stratification $X = \bigsqcup_{w \in W} X_w$ we need to show that the Schubert stratification is Whitney-linear.

Except for condition (i), the Whitney-linear conditions are obvious. Condition (i) is verified in [SW, Proposition C.2]. Let me sketch a different proof. We need to show that $i_v^* \nabla_w \in DMT(X_v)$. Let X^v be the Schubert cell opposite to X_v. Then using the standard technique involving 'contracting slices' (see [Sp, Corollary 2]), we obtain that $i_v^* \nabla_w \simeq a_* \underline{X_v \times (X_w \cap X^v)}$ (up to shift), where $a\colon X_v \times (X_w \cap X^v) \to X_v$ is the evident projection. Now it is well known (and easy to see) that the $X_w \cap X^v$ are linear varieties.

References

[B] A. Beilinson, *Remarks on Grothendieck's standard conjectures*, arXiv:1006.1116v2.

[BGS] A. Beilinson, V. Ginzburg, W. Soergel, *Koszul duality patterns in Representation Theory*, J.A.M.S. **9** (1996), 473-526.

[CD] D-C. Cisinski, F. Déglise, *Triangulated categories of mixed motives*, arXiv:0912.2110.

[L] M. Levine, *Tate motives and the vanishing conjectures for algebraic K-theory*, Algebraic K-theory and Algebraic Topology (P.G. Goerss and J.F. Jardine eds.) Kluwer Acad. Publ. Dordrecht (1993), NATO ASI Series C, Vol. 407, 167-188.

[SW] W. Soergel, M. Wendt, *Perverse motives and graded category $\mathcal{O}$*, arXiv:1404.6333v1.

[Sp] T.A. Springer, *A purity result for fixed point varieties in flag manifolds*, J. Fac. Sci., Univ. Tokyo, Sect. 1A **31** (1984), 231-238.

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The Appalachians