EQUIVARIANT NEARBY CYCLES

R. VIRK

Last updated: 11 December, 2014

1. **Preliminaries.** We play with complex varieties. Let G be an algebraic group and X a G-variety. A diagram

$$X \xleftarrow{p} X_G \xrightarrow{q} \bar{X}$$

with p equivariant and q a G-torsor will be called a *resolution* of X. If p is smooth (resp. n-acyclic) we will call it a smooth (resp. n-acyclic) resolution. We will say that $\bar{M} \in D\bar{X}$ *comes from* X if there exists some $M \in DX$ along with an isomorphism

$$p^*M \simeq q^*\bar{M}.$$

Let $f\colon G \to \mathbf{A}^1$ be a *G-invariant* morphism. That is, $f(g \cdot x) = f(x)$ for all $g \in G$ and $x \in X$. Then we have a diagram

$$\begin{array}{ccccc} X^* & \longrightarrow & X & \longleftarrow & X_0 \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ \mathbf{A}^1 - \{0\} & \longrightarrow & \mathbf{A}^1 & \longleftarrow & \{0\} \end{array}$$

with all squares cartesian (the lower horizontal maps being the evident inclusions). The morphism f induces a morphism $\bar{f}\colon \bar{X} \to \mathbf{A}^1$. So we obtain a diagram

$$\begin{array}{ccccc} \bar{X}^* & \longrightarrow & \bar{X} & \longleftarrow & \bar{X}_0 \\ \downarrow & & \downarrow{\scriptstyle \bar{f}} & & \downarrow \\ \mathbf{A}^1 - \{0\} & \longrightarrow & \mathbf{A}^1 & \longleftarrow & \{0\} \end{array}$$

with all squares cartesian. This yields the nearby cycles functor

$$\psi_{\bar{f}}\colon D\bar{X} \to D\bar{X}_0.$$

Lemma 1.1. *Assume $\bar{M} \in D\bar{X}$ comes from DX. Then $\psi_{\bar{f}}(\bar{M})$ comes from DX_0.*

Proof. By assumption, there exists $M \in DX$ along with an isomorphism

$$p^*M \simeq q^*\bar{M}.$$

As taking nearby cycles commutes with pulling back along smooth morphisms,

$$p^*\psi_f(M) \simeq \psi_{f\circ p}(p^*M) \simeq \psi_{f\circ p}(q^*\bar{M}) \simeq q^*\psi_f(\bar{M}). \qquad \square$$

Theorem 1.2. *Assume that X admits n-acyclic smooth resolutions for each n. Then the nearby cycles functor $DX \to DX_0$ lifts to the equivariant setting. That is, we have a functor, the* equivariant nearby cycles,

$$\psi_f \colon D_G X \to D_G X_0$$

compatible with nearby cycles on DX which satisfies all the usual properties of the ordinary nearby cycles.

Proof. Let $M \in D_G X$. The previous Lemma yields an object $\psi_{\bar{f}} \bar{M} \in D\bar{X}_0$ for each resolution $X_0 \leftarrow E \to \bar{X}_0$ that comes from X. These objects are compatible under smooth pullback, since the nearby cycles functor is. As X admits n-acyclic resolutions, this prescription extends to yield a functor

$$\psi_f \colon D_G X \to D_G X_0.$$

It is clear that this functor has all the required properties. □

Proposition 1.3. *Let $H \subseteq G$ be a closed subgroup. Then ψ_f commutes with the restriction functor* res_H^G*. Moreover, if G/H is complete, then ψ_f also commutes with induction from H to G.*

Proof. Restriction is given by pulling back along a smooth morphism. So commutation of ψ_f with res_H^G is immediate. Similarly, induction is given by a push forward ($*$- or !- depending on whether we want a right or left adjoint[1] to res_H^G). If G/H is complete, then this push forward occurs along a proper morphism. This yields the desired result, since nearby cycles commute with push forwards along proper morphisms. □

References

[BL] J. Bernstein, V. Lunts, *Equivariant sheaves and functors*, Lecture Notes in Math. **1578**, Springer-Verlag, Berlin (1994).

[1] In the case of the !-pushforward one should also incorporate a shift in order to get a left adjoint to res_H^G. However, this is irrelevant to the discussion at hand.