ON FIXED POINT STACKS

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1. Introduction. This note observes that the fixed point (1-)stack of a finite group action may be computed correctly at the level of a presentation by groupoids (such as that afforded by an atlas). That is, *before* stackification. Related results such as the compatibility of fixed points with fibre products, intertia stacks and non-abelian cohomology are also discussed. For simplicity, only sites with enough points are considered. For arbitrary sites the same reasoning should go through using Boolean localization. However, I have not checked the details of this.

2. Conventions. 'Map', 'morphism' and 'arrow' are used interchangeably. Maps are 'canonical' if they are natural transformations of functors. A map between categories means a functor of categories. The symbol $*$ is reserved for the final object in a category (if it exists). The symbol $\cong$ is only used for isomorphisms. Weaker notions of equivalence (such as equivalence of categories, weak equivalence, etc.) are denoted by $\simeq$ or $\xrightarrow{\sim}$. A groupoid is a small category in which all morphisms are invertible. For homotopy theoretic notions, simplicial sets are conflated with topological spaces. Throughout, Γ denotes a *finite* group.

3. Homotopy invariance. Let X be a groupoid equipped with a Γ-action. That is, Γ acts on the sets of objects and morphisms in a manner compatible with the structure maps of X as a category. Let $\mathbb{E}\Gamma$ be the groupoid with set of objects Γ, and a unique morphism between any two objects. Then Γ acts on $\mathbb{E}\Gamma$ in the obvious way. Let

$$X^{h\Gamma} = \mathrm{Map}_{\Gamma}(\mathbb{E}\Gamma, X)$$

be the groupoid of Γ-equivariant functors and natural transformations $\mathbb{E}\Gamma \to X$. An object of $X^{h\Gamma}$ amounts to a pair (x,ϕ), where x is an object of X and ϕ is a rule that assigns, to each $g \in \Gamma$, a morphism $\phi(g)\colon x \to gx$ in X, satisfying $\phi(1) = \mathrm{id}$ and

$$\phi(gh) = g\phi(h) \circ \phi(g), \tag{3.0.1}$$

for all $g, h \in \Gamma$. An arrow $(x,\phi) \to (x_1,\phi_1)$ is a morphism $\alpha\colon x \to x_1$ in X such that

$$\phi_1(g) \circ \alpha = g\alpha \circ \phi(g), \tag{3.0.2}$$

for all $g \in \Gamma$. If $f : X \to Y$ is an equivariant map between groupoids equipped with Γ-actions, then we obtain a morphism of groupoids $f^{h\Gamma}\colon X^{h\Gamma} \to Y^{h\Gamma}$ via

$$f^{h\Gamma}((x,\phi)) = (f(x), g \mapsto f(\phi(g))), \qquad f^{h\Gamma}(\alpha) = f(\alpha).$$

Example 3.1. For the trivial action, $X^{h\Gamma}$ consists of $x \in X$ along with a Γ-action on x.

Example 3.2. The ordinary Γ-fixed points of X are not invariant under equivalence of categories: $\mathbb{E}\Gamma$ is equivalent to $*$, but the Γ-action on $\mathbb{E}\Gamma$ is free.

Theorem 3.3. *If f is an equivalence of categories, then so is $f^{h\Gamma}$.*

First proof. Let B denote the nerve functor from the category of small categories to the category of simplicial sets. Then B is full, faithful and commutes with products. Consequently, we obtain a canonical isomorphism

$$B\mathrm{Map}_\Gamma(\mathbb{E}\Gamma, X) \xrightarrow{\cong} \mathrm{Map}_\Gamma(E\Gamma, BX),$$

where the right hand side is the (equivariant) simplicial function complex and $E\Gamma = B\mathbb{E}\Gamma$ (see [G, Section 4.4] or [T, Section 2]). Now $E\Gamma$ is contractible and Γ acts freely on it. As BX and BY are Kan complexes, this implies

$$Bf^{h\Gamma}\colon \mathrm{Map}_\Gamma(E\Gamma, BX) \to \mathrm{Map}_\Gamma(E\Gamma, BY)$$

is a weak equivalence of topological spaces (see [DW, Section 10] for a concise exposition of these topological properties). This yields the desired result. □

Second proof. We argue that the functor $f^{h\Gamma}$ is full, faithful and essentially surjective. Start with faith. It suffices to show that $f^{h\Gamma}$ is injective on automorphism groups. This is immediate from the definitions.

Now we show $f^{h\Gamma}$ is full. Let (x,ϕ) and (x_1,ϕ_1) be in $X^{h\Gamma}$, and let $\beta\colon f(x) \to f(x_1)$ be a morphism in Y corresponding to an arrow $(f(x), f(\phi)) \to (f(x_1), f(\phi_1))$ in $Y^{h\Gamma}$. As f is full, there exists $\alpha\colon x \to x_1$ such that $f(\alpha) = \beta$. Further,

$$\begin{aligned} f(\phi_1(g)\circ\alpha) &= f(\phi_1(g))\circ\beta && (\text{as } f(\alpha)=\beta)\\ &= g\beta\circ f(\phi(g)) && (\text{by } (3.0.2))\\ &= gf(\alpha)\circ f(\phi(g)) && (\text{as } f(\alpha)=\beta)\\ &= f(g\alpha\circ\phi(g)) && (\text{since } f \text{ is equivariant}). \end{aligned}$$

As f is faithful, this implies $\phi_1(g)\circ\alpha = g\alpha\circ\phi(g)$, for all $g\in\Gamma$. That is, α defines a map $(x,\phi)\to(x_1,\phi_1)$ in $X^{h\Gamma}$. By construction, this arrow is mapped by $f^{h\Gamma}$ to the morphism $(f(x), f(\phi)) \to (f(x_1), f(\phi_1))$ that we started with. Hence, $f^{h\Gamma}$ is full.

To show $f^{h\Gamma}$ is essentially surjective, let $(y,\psi)\in Y^{h\Gamma}$. As f is essentially surjective, there exists an isomorphism $\alpha\colon f(x)\to y$, for some $x\in X$. For $g\in\Gamma$, set

$$\gamma(g) = g\alpha^{-1}\circ\psi(g)\circ\alpha.$$

Then $\gamma(1) = \mathrm{id}$ and

$$\begin{aligned} \gamma(gh) &= gh\alpha^{-1}\circ\psi(gh)\circ\alpha && (\text{by definition of } \gamma)\\ &= gh\alpha^{-1}\circ g\psi(h)\circ\psi(g)\circ\alpha && (\text{by } (3.0.1))\\ &= g(h\alpha^{-1}\circ\psi(h)\circ\alpha)\circ(g\alpha^{-1}\circ\psi(g)\circ\alpha) && (\text{as } g\alpha\circ g\alpha^{-1} = g(\mathrm{id}) = \mathrm{id})\\ &= g\gamma(h)\circ\gamma(g) && (\text{by definition of } \gamma), \end{aligned}$$

for all $g,h\in\Gamma$. Hence, $(f(x),\gamma)\in Y^{h\Gamma}$. As $\gamma(g)$ is a morphism $f(x)\to f(gx)$ and f is full, there exists $\phi(g)\colon x\to gx$ such that $f(\phi(g)) = \gamma(g)$ for every $g\in\Gamma$. As f is faithful, we must have $\phi(gh) = g\phi(h)\circ\phi(g)$. Thus, we obtain an object $(x,\phi)\in X^{h\Gamma}$ that is mapped by f to $(f(x),\gamma)$. Further, $\psi(g)\circ\alpha = g\alpha\circ\gamma(g)$. So α defines an isomorphism $(f(x),\gamma)\xrightarrow{\cong}(y,\psi)$. In other words, $f^{h\Gamma}$ is essentially surjective. □

Example 3.4. Let G be an arbitrary group on which Γ acts via automorphisms. Let $Z^1(\Gamma; G)$ be the set of G-valued 1-cocycles of Γ. I.e., functions $\sigma : \Gamma \to G$ satisfying

$$\sigma(gh) = \sigma(g) \cdot g\sigma(h)$$

for all $g, h \in \Gamma$. Let $\mathbb{E}_G Z^1(\Gamma; G)$ be the groupoid with set of objects $Z^1(\Gamma; G)$, and an arrow $\sigma \to \sigma_1$ for each $\alpha \in G$ such that

$$\sigma_1(g) = \alpha \cdot \sigma(g) \cdot g\alpha^{-1},$$

for all $g \in \Gamma$. The set of isomorphism classes of objects in $\mathbb{E}_G Z^1(\Gamma; G)$ is the first (non-abelian) cohomology $H^1(\Gamma; G)$ (see [S, Chapter 1.5]). The automorphism group of $\sigma \in \mathbb{E}_G Z^1(\Gamma; G)$ is

$$K_\sigma = \{\alpha \in G \mid \sigma(g) \cdot g\alpha \cdot \sigma(g)^{-1} = \alpha \text{ for all } g \in \Gamma\}.$$

Thus, there is an equivalence of categories

$$\mathbb{E}_G Z^1(\Gamma; G) \simeq \coprod_{[\sigma] \in H^1(\Gamma; G)} \mathbb{B}K_\sigma,$$

where $\mathbb{B}K_\sigma$ denotes the groupoid consisting of a single object with automorphisms K_σ. This equivalence depends on picking an object in each isomorphism class $[\sigma]$. In particular, there is usually no way to make it canonical.

Let $\mathbb{B}G$ be the groupoid consisting of a single object with automorphism group G. Then Γ acts on $\mathbb{B}G$. Objects in $(\mathbb{B}G)^{h\Gamma}$ amount to functions $\phi : \Gamma \to G$ satisfying $\phi(1) = 1$ and (3.0.1). To such a function, assign a cocycle $\sigma_\phi \in Z^1(\Gamma; G)$ by setting $\sigma_\phi(g) = \phi(g)^{-1}$, for all $g \in \Gamma$. This yields a canonical isomorphism

$$(\mathbb{B}G)^{h\Gamma} \xrightarrow{\cong} \mathbb{E}_G Z^1(\Gamma; G).$$

Hence, we obtain an equivalence of categories

$$(\mathbb{B}G)^{h\Gamma} \simeq \coprod_{[\sigma] \in H^1(\Gamma; G)} \mathbb{B}K_\sigma.$$

In principle, this gives a presentation of $X^{h\Gamma}$ for an arbitrary groupoid X, since any groupoid is equivalent to one of the form $\coprod_i \mathbb{B}G_i$. As before, this involves picking objects in isomorphism classes and usually cannot be made canonical.

4. **Filtered colimits.** All small limits and colimits exist in the category of groupoids (see [Hig, Chapter 7,9] or [Hol, Appendix A]). Limits are given by limits of sets on objects and morphisms. Colimits can be nebulous to describe explicitly. However, filtered colimits present no difficulties. Filtered colimits of groupoids are given by filtered colimits of sets on objects and morphisms.

Let $i \mapsto X_i$ be a functor from a small category I to groupoids with Γ-action. In particular: (a) each groupoid X_i is equipped with a Γ-action; (b) for each morphism $i \to j$ in I, the corresponding map $X_i \to X_j$ is Γ-equivariant.

Lemma 4.1. *If I is filtered and Γ is a finite group, then the canonical map of groupoids*

$$\varinjlim_I X_i^{h\Gamma} \to (\varinjlim_I X_i)^{h\Gamma}$$

is an isomorphism.

Proof. The inverse is given as follows. Let $(x,\phi) \in (\varinjlim_I X_i)^{h\Gamma}$. Then there exists an object x_i in some X_i, and maps $\phi_j(g)$ in various X_{j_g} that satisfy the defining relations (**3.0.1**), in various $X(k_{(g,h)})$, such that x is the image of x_i, and the maps $\phi(g)$ are the image of the $\phi_j(g)$. As I is filtered and there are only finitely many of these relations, we may assume $i = j_g = k_{(g,h)}$ for all $g \in \Gamma$ and $(g,h) \in \Gamma \times \Gamma$. This yields an object in $(\varinjlim_I X_i)^{h\Gamma}$ that does not depend on the choices made. Similarly, any morphism in $(\varinjlim_I X_i)^{h\Gamma}$ also yields a well defined arrow in $\varinjlim_I X_i^{h\Gamma}$. □

Example 4.2. The heart of the matter in Lemma 4.1 is that the nerve of a groupoid is a 2-coskeletal Kan complex. Consequently, homotopy limits (in the sense of [BK]) of finite diagrams of groupoids behave as if they were over very small categories (a category is very small if its nerve has finitely many non-degenerate simplices). The naïve analogue of Lemma 4.1 for general simplicial sets (replacing equivalence of categories by weak equivalence of spaces), even Kan complexes, is false. Here is a standard counterexample.

Let $B\Gamma$ be the classifying space of Γ. For a topological space X equipped with the trivial Γ-action, $X^{h\Gamma} = \mathrm{Map}(B\Gamma, X)$. This can be taken as a definition (the right hand side denotes the set of continuous maps $B\Gamma \to X$ equipped with the compact open topology). Let $\mathrm{sk}_n(B\Gamma)$ denote the n-skeleton of $B\Gamma$. The identity map on $B\Gamma$ does not factor through any finite skeleton. Consequently, $\varinjlim_n \mathrm{Map}(B\Gamma, \mathrm{sk}_n(B\Gamma))$ and $\mathrm{Map}(B\Gamma, B\Gamma)$ cannot be weakly equivalent.

Regardless, one may prove a generalization of Lemma 4.1 for n-coskeletal spaces. This approach is more conceptual, but not as elementary as the proof given above.

5. **Presheaves of groupoids.** Let $\mathscr{C}$ be a Grothendieck site with enough points. Set

$$\mathrm{PreGrpd}(\mathscr{C}) = \text{category of presheaves of groupoids on } \mathscr{C}.$$

Note: 'presheaf' means 'strict presheaf' (as opposed to lax). We will not distinguish between an object $X \in \mathscr{C}$ and the presheaf $\mathrm{Hom}_{\mathscr{C}}(-, X)$ it represents.

A stalk for an object of $\mathrm{PreGrpd}(\mathscr{C})$ is defined using the same formula as for presheaves of sets. Each such stalk is a groupoid. A map in $\mathrm{PreGrpd}(\mathscr{C})$ is called a *local weak equivalence* if it induces an equivalence of categories (groupoids) on all stalks. A map $X \to Y$ in $\mathrm{PreGrpd}(\mathscr{C})$ is a *sectionwise weak equivalence* if it induces an equivalence of categories $X(U) \to Y(U)$ for all $U \in \mathscr{C}$. A sectionwise weak equivalence is always a local weak equivalence. However, the converse does not generally hold.

Example 5.1. A morphism between sheaves of sets (viewed as groupoids) is a local weak equivalence if and only if it is an isomorphism.

Example 5.2. If $\mathscr{C} = *$, then $\mathrm{PreGrpd}(\mathscr{C})$ is the category of groupoids. A local weak equivalence amounts to an equivalence of categories.

Example 5.3. Let $\mathscr{C}$ be the big étale site on complex varieties.[1] Let G be a linear algebraic group. Define $BG \in \mathrm{PreGrpd}(\mathscr{C})$ by taking $BG(U)$ to consist of a single object with automorphism group $G(U)$. Let $[BG]$ denote the classifying stack of

[1]'Complex variety' = 'separated scheme over Spec($\mathbf{C}$)'.

principal G-bundles. The map $BG \to [BG]$ sending the single object in $BG(U)$ to the trivial bundle on U is a local weak equivalence (étale local triviality of principal bundles). However, it is not a sectionwise weak equivalence.

Suppose a finite group Γ acts on $X \in \mathrm{PreGrpd}(\mathscr{C})$. That is, Γ acts on each $X(U), U \in \mathscr{C}$, and these actions are compatible with restriction maps. Define $X^{h\Gamma}$ by

$$X^{h\Gamma}(U) = X(U)^{h\Gamma}.$$

An equivariant map $f : X \to Y$ between presheaves of groupoids equipped with Γ-actions induces a map $f^{h\Gamma} : X^{h\Gamma} \to Y^{h\Gamma}$.

Proposition 5.4. *If f is a local weak equivalence, then so is $f^{h\Gamma}$.*

Proof. By definition of a point of a site, the stalk of a sheaf of sets at a point is a left exact functor to the category of sets. On the other hand, stalks are defined as colimits (over small categories) in the category of sets [Stacks, Tag 00Y3]. A small category I is filtered if and only if all colimits into the category of sets, indexed by I, commute with finite limits [KS, Theorem 3.1.6]. Consequently, stalks are given by filtered colimits. Hence, the result follows from Lemma 4.1 and Theorem 3.3. □

6. Derived functors. Let $\mathrm{PreGrpd}(\mathscr{C})^\Gamma$ be the category of $X \in \mathrm{PreGrpd}(\mathscr{C})$ equipped with a Γ-action (morphisms given by equivariant maps). Let $\varprojlim_\Gamma X$ be the ordinary fixed points of X (i.e., take Γ-fixed points on sets of objects and morphisms in each $X(U), U \in \mathscr{C}$). Then both $\varprojlim_\Gamma$ and $(-)^{h\Gamma}$ define functors

$$\mathrm{PreGrpd}(\mathscr{C})^\Gamma \to \mathrm{PreGrpd}(\mathscr{C}).$$

The functor $\varprojlim_\Gamma$ does not preserve local weak equivalences (Example 3.2).

Let $\mathrm{Ho}(\mathrm{PreGrpd}(\mathscr{C}))$ (resp. $\mathrm{Ho}(\mathrm{PreGrpd}(\mathscr{C})^\Gamma)$) be the localization of $\mathrm{PreGrpd}(\mathscr{C})$ (resp. $\mathrm{PreGrpd}(\mathscr{C})^\Gamma$) with respect to local weak equivalences (resp. Γ-equivariant local weak equivalences). Let

$$\mathbf{R}\varprojlim_\Gamma : \mathrm{Ho}(\mathrm{PreGrpd}(\mathscr{C})^\Gamma) \to \mathrm{Ho}(\mathrm{PreGrpd}(\mathscr{C}))$$

denote the total right derived functor of $\varprojlim_\Gamma$, if it exists, in the sense of [Q, Chapter I.4]. By Proposition 5.4, $(-)^{h\Gamma}$ induces a functor $\mathrm{Ho}(\mathrm{PreGrpd}(\mathscr{C})^\Gamma) \to \mathrm{Ho}(\mathrm{PreGrpd}(\mathscr{C}))$.

Theorem 6.1. *The right derived functor $\mathbf{R}\varprojlim_\Gamma$ exists. Moreover, there is a canonical isomorphism $\mathbf{R}\varprojlim_\Gamma X \cong X^{h\Gamma}$ in $\mathrm{Ho}(\mathrm{PreGrpd}(\mathscr{C}))$.*

Proof. Existence follows from the fact that local weak equivalences are the class of weak equivalences in a simplicial model structure on $\mathrm{PreGrpd}(\mathscr{C})$ (the so called injective model structure, see [Hol, Theorem 1.4]; [J, Chapter 9.2] is also a convenient reference). This induces a model structure on $\mathrm{PreGrpd}(\mathscr{C})^\Gamma$ that allows for the construction of the right derived functor using [Q, Chapter I.4] (see [BK, Chapter XI, Section 8] and [Hir, Chapter 18]). In fact, since the simplicial structure on $\mathrm{PreGrpd}(\mathscr{C})$ is defined sectionwise, an explicit model for $\mathbf{R}\varprojlim_\Gamma X$ is given as follows. Let $X \to RX$ be a functorial fibrant model of X. In partcular, RX is fibrant and $X \to RX$ is a local weak equivalence. The Γ-action on X induces an action on RX for which this local weak equivalence is equivariant. Now $\mathbf{R}\varprojlim_\Gamma X = (RX)^{h\Gamma}$. Thus, Proposition 5.4 yields the desired result. □

7. **Stacks.** An object in $\mathrm{PreGrpd}(\mathscr{C})$ is a stack if it is a sheaf of groupoids that satisfies effective descent. A map between stacks is a local weak equivalence if and only if it is a sectionwise weak equivalence [J, Proposition 9.2.8]. In other words, local weak equivalence for stacks amounts to the usual notion of equivalence of stacks. For a stack X equipped with the action of a finite group Γ, the definition of $X^{h\Gamma}$ amounts to that of fixed point stack in [R].

Proposition 7.1. *If X is a stack, then so is $X^{h\Gamma}$.*

Proof. Use the (injective) model structure on $\mathrm{PreGrpd}(\mathscr{C})$ from the previous proof. Let $X \xrightarrow{\sim} RX$ be a fibrant model for X. Then $X^{h\Gamma} \xrightarrow{\sim} (RX)^{h\Gamma}$ is a fibrant model for $X^{h\Gamma}$ by Theorem 6.1 and [Hir, Theorem 18.5.2 (2)]. Moreover, fibrant objects are stacks [J, Remark 9.2.3]. So, if X is a stack, $X \to RX$ is a sectionwise weak equivalence, and Theorem 3.3 implies $X^{h\Gamma} \to (RX)^{h\Gamma}$ is also a sectionwise weak equivalence. □

Every $X \in \mathrm{PreGrpd}(\mathscr{C})$ admits a canonical stack completion $[X]$ (often called the stackification of X, see [Stacks, Tag 02ZO] or [J, Chapter 9.2]). I.e., there is a canonical local weak equivalence $X \xrightarrow{\sim} [X]$ with $[X]$ a stack. We have already encountered these implicitly in the proofs above via fibrant models.

Corollary 7.2. *The stacks $[X]^{h\Gamma}$ and $[X^{h\Gamma}]$ are canonically equivalent.*

8. **Pullbacks and inertia.** Let $f : X \to Z$ and $g : Y \to Z$ be maps of groupoids. The homotopy pullback of the diagram $X \xrightarrow{f} Z \xleftarrow{g} Y$ is the groupoid

$$X \times_Z^h Y = \{(x, y, \phi) \mid x \in X, y \in Y, \phi \in \mathrm{Hom}(f(x), g(y))\},$$

with morphisms defined by the evident commutative diagrams. One extends this definition to $\mathrm{PreGrpd}(\mathscr{C})$ by taking the above to be a sectionwise prescription. Note that $X \times_Z^h Y$ is the (2-)fibre product of [Stacks, Tag 0030].

We will not use this explicitly, but it is comforting to know that the homotopy pullback is the total right derived functor of the usual (strict) fibre product. However, we do need the following homotopy invariance (the so-called 'co-glueing Lemma'): if $X \to X_1$, $Y \to Y_1$ and $Z \to Z_1$ are local weak equivalences such that the diagram

$$\begin{array}{ccccc} X & \longrightarrow & Z & \longleftarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ X_1 & \longrightarrow & Z_1 & \longleftarrow & Y_1 \end{array}$$

commutes (on the nose), then the evident canonical map $X \times_Z^h Y \to X_1 \times_{Z_1}^h Y_1$ is also a local weak equivalence. For a direct proof see [Stacks, Tag 02XB].

If X, Y, Z are equipped with Γ-actions and f, g are equivariant, then there is an induced Γ-action on $X \times_Z^h Y$. One also has a canonical map $(X \times_Z^h Y)^{h\Gamma} \to X^{h\Gamma} \times_{Z^{h\Gamma}}^h Y^{h\Gamma}$.

Proposition 8.1. *The canonical map*

$$(X \times_Z^h Y)^{h\Gamma} \to X^{h\Gamma} \times_{Z^{h\Gamma}}^h Y^{h\Gamma}$$

is a local weak equivalence.

Proof. Apply Theorem 3.3 and the aforementioned homotopy invariance. □

For a groupoid X, the free loop groupoid of X is

$$LX = \{(x, \phi) \mid x \in X, \phi \in \mathrm{Aut}(x)\},$$

with morphisms given by the obvious commutative diagrams. As before, one extends this definition to PreGrpd($\mathscr{C}$) by taking this to be a sectionwise prescription. There is a canonical local weak equivalence [Stacks, Tag 034H]:

$$LX \xrightarrow{\sim} X \times^h_{X \times X} X,$$

where the pullback is over the diagonal map $X \to X \times X$. When X is a stack, LX is often called the inertia stack of X, and is denoted IX (sometimes ΛX) in the literature. If Γ acts on X, then there is an induced action on LX.

Corollary 8.2. *There is a canonical local weak equivalence* $(LX)^{h\Gamma} \xrightarrow{\sim} L(X^{h\Gamma})$.

Example 8.3. Let $\mathscr{C}$ be the big étale site of complex varieties. Let Γ be a finite group acting on a linear algebraic group G via automorphisms. We will use the notation of Example 3.4 and Example 5.3. In particular, $BG(U)$ denotes the groupoid consisting of a single object with automorphisms $G(U)$, for all $U \in \mathscr{C}$; $[BG]$ is its stack completion - the stack of principal G-bundles; $Z^1(\Gamma; G(U))$ is the set of 1-cocycles, etc. Then Γ acts on BG, and we have a canonical isomorphism

$$(BG)^{h\Gamma} \xrightarrow{\cong} E_G Z^1(\Gamma; G),$$

where, for $U \in \mathscr{C}$, in the notation of Example 3.4,

$$E_G Z^1(\Gamma; G)(U) = \mathbb{E}_{G(U)} Z^1(\Gamma; G(U)).$$

As Γ is finite, and in particular has a finite presentation, the presheaf $U \mapsto Z^1(\Gamma; G(U))$ is representable by an affine G-variety Z. Explicitly, Z is the variety of group homomorphisms $\Gamma \to G \rtimes \Gamma$ that are sections of the canonical map $G \rtimes \Gamma \to \Gamma$.

Embed $G \rtimes \Gamma$ in some GL_n and consider the representation variety Y of group homomorphisms $\Gamma \to GL_n$. Every representation of Γ over $\mathbf{C}$ is completely reducible. Further, for a fixed dimension there are finitely many isomorphism classes of these. So, under the conjugation action of GL_n, the variety Y has finitely many orbits and each orbit is closed [PR, Theorem 2.17]. On the other hand, Z embeds into Y, and each G-orbit in Z is the intersection of some GL_n-orbit with Z. Thus, there are finitely many G-orbits in Z and they are all closed (see [PR, Lemma 2.11 and Theorem 2.17]). Consequently, Z is isomorphic to the disjoint union of its orbits. These orbits are parametrized by $H^1(\Gamma; G(\mathbf{C}))$ (see Example 3.4). Moreover, for a geometric point $\sigma \in Z^1(\Gamma; G(\mathbf{C}))$, the stabilizer of σ is

$$K_\sigma = \{\alpha \in G \mid \sigma(g) \cdot g\alpha \cdot \sigma(g)^{-1} = \alpha \text{ for all } g \in \Gamma\}.$$

Thus, we have a local weak equivalence

$$(BG)^{h\Gamma} \simeq \coprod_{[\sigma] \in H^1(\Gamma; G(\mathbf{C}))} BK_\sigma.$$

As in Example 3.4, it is generally impossible to make this canonical.

Write $[Z/G]$ for the stack completion of $E_G Z^1(\Gamma; G)$, i.e., the stack quotient of Z by G. By Corollary 7.2, we have equivalences of stacks:

$$[BG]^{h\Gamma} \simeq [Z/G] \simeq \coprod_{[\sigma]\in H^1(\Gamma; G(\mathbf{C}))} [BK_\sigma].$$

The first of these equivalences is canonical. However, the second depends on the various choices made above and cannot usually be made canonical. In particular, it is generally impossible to identify the first and last spaces here.

Readers may explore the consequences of Proposition 8.1 and Corollary 8.2 in this setting at their own leisure.

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The Appalachians