HOMOTOPY LIMITS AND FIXED POINT STACKS

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We discuss homotopy limits of stacks in groupoids. These are sometimes called lax limits, 2-limits, or derived limits in the literature. Inertia stacks, the usual stack fibre product and the fixed point stacks are all examples of these. We examine the compatibility of homotopy limits with each other in general, as well as with other functorial constructions such as stackification.

For instance, we show (Corollary 6.10) that in the case of a finite diagram, homotopy limits may be computed correctly using a presentation by groupoids (such as that afforded by an atlas). I.e., *before* stackification. In particular, this applies to fixed point stacks (for finite group actions) and subsumes known results about (2-)fibre products and inertia stacks. The reader should glance at Example 6.12 for a taste of this perspective in action. Skimming the statements of Theorem 6.4 through Theorem 6.11 will provide a reasonable overview of results.

For simplicity, only sites with enough points are considered. For arbitrary sites, the same arguments should go through using Boolean localization. However, I have not checked the details of this.

Let me emphasize that the essential ideas around homotopy limits are all extracted from the ur-reference [BK]. I merely translate[1] some sections of [BK] to the *much simpler* special case of groupoids and make some observations that are specific to the latter (for instance, see Lemma 4.2 and the remarks after it about the 'canonical mistake'). The connection with stacks seems to have been first made explicit in [Hol], but goes back to at least [Jo]. I personally found [A], [J] and [Te] enlightening while connecting the dots. The précis: readers looking for originality should stop here and peruse the aforementioned works instead.

1. **Conventions.** 'Map', 'morphism' and 'arrow' are used interchangeably. 'Canonical' is written in lieu of 'a natural transformation of functors'. A map of categories means a functor of categories. The symbol $*$ is reserved for the final object in a category (if it exists). The symbol $\cong$ is only used for isomorphisms. Weaker notions of equivalence (equivalence of categories, weak equivalence, etc.) are denoted by $\simeq$ or $\xrightarrow{\sim}$.

2. **Groupoids.** A groupoid is a small category in which all maps are invertible. Set

$$\mathrm{Grpd} = \text{category of groupoids.}$$

To emphasize, Grpd is a strict (1-)category (as opposed to a 2-category). A map of groupoids $f: X \to Y$ is called a *weak equivalence* if it is an equivalence of categories. The map f is called a *fibration* if, for each $x \in X$ and each isomorphism $\alpha\colon f(x) \to y$

[1] probably quite badly and with several failings in my understanding

in Y, there exists an isomorphism $x \to x_1$ in X, such that $f(\beta) = \alpha$. A fibration that is also a weak equivalence is called *acyclic*

Example 2.1. Let G be a group and let X be a G-set. Write $\mathbb{E}_G X$ for the groupoid with set of objects X and an arrow $x \xrightarrow{g} y$ for each $g \in G$ such that $g(x) = y$. A map of G-sets $X \to Y$ (i.e., an equivariant map) induces a map $\mathbb{E}_G X \to \mathbb{E}_G Y$. If $X \to Y$ is an isomorphism of G-sets, then $\mathbb{E}_G X \to \mathbb{E}_G Y$ is an isomorphism of groupoids. If $N \subset G$ is a normal subgroup that acts on X freely, then the evident canonical map $\mathbb{E}_G X \to \mathbb{E}_{G/N} X/N$ is an acyclic fibration. Set

$$\mathbb{E}G = \mathbb{E}_G G \qquad \text{and} \qquad \mathbb{B}G = \mathbb{E}_G\{*\}.$$

I.e., the groupoids associated to G acting on itself and the set $\{*\}$, respectively.

3. **Homotopy limits: groupoids.** If K and G are small categories, set

$$\mathrm{Map}(K, G) = \text{category of functors } K \to G.$$

Morphisms are natural transformations of functors. The obvious canonical map

$$\mathrm{Map}(K \times H, G) \to \mathrm{Map}(K, \mathrm{Map}(H, G))$$

is an isomorphism (the so-called exponential law).

If Γ is a small category and α is an object of Γ, then the overcategory $(\Gamma \downarrow \alpha)$ is the category whose objects are morphisms $\beta \to \alpha$ in Γ. Maps are given by the obvious commutative triangles. Let Γ be a small category. Let $X : \Gamma \to \mathrm{Grpd}$, $\alpha \mapsto X_\alpha$ be a functor. In other words, X is a Γ-diagram of groupoids. The *homotopy limit* of X, denoted $\underleftarrow{\mathrm{holim}}_\Gamma X$, is the equalizer of the maps

$$\prod_{\alpha \in \Gamma} \mathrm{Map}((\Gamma \downarrow \alpha), X_\alpha) \underset{\psi}{\overset{\phi}{\rightrightarrows}} \prod_{\{\alpha \to \beta\} \in \Gamma} \mathrm{Map}((\Gamma \downarrow \alpha), X_\beta),$$

where ϕ is induced by the natural maps

$$\mathrm{Map}((\Gamma \downarrow \alpha), X_\alpha) \to \mathrm{Map}((\Gamma \downarrow \alpha), X_\beta)$$

and ψ is induced by the canonical maps

$$\mathrm{Map}((\Gamma \downarrow \beta), X_\beta) \to \mathrm{Map}((\Gamma \downarrow \alpha), X_\beta).$$

As all small limits exist in the category of groupoids [Hig, Chapter 7] (they are given by limits of sets on objects and morphisms), this equalizer always exists.

This is a functorial construction: if $f : X \to Y$ is a natural transformation of Γ-diagrams, then we have an induced map

$$f_* \colon \underleftarrow{\mathrm{holim}}_{\Gamma} X \to \underleftarrow{\mathrm{holim}}_{\Gamma} Y.$$

Note that the usual limit of the Γ-diagram X is the equalizer of

$$\prod_{\alpha \in \Gamma} X_\alpha \rightrightarrows \prod_{\{\alpha \to \beta\} \in \Gamma} X_\beta,$$

where the maps are the analogues of ϕ and ψ above. This may be rewritten as

$$\prod_{\alpha\in\Gamma}\mathrm{Map}(*,X_\alpha) \rightrightarrows \prod_{\{\alpha\to\beta\}\in\Gamma}\mathrm{Map}(*,X_\beta).$$

Thus, we also have a canonical map $\varprojlim_\Gamma X \to \operatorname{holim}_{\longleftarrow\,\Gamma} X$. In general, this map is not an isomorphism or even a weak equivalence.

Example 3.1. Let Γ be the pullback category. I.e., Γ has three objects x, y, z, and non-identity morphisms depicted by the diagram $x \to z \leftarrow y$. A functor $\Gamma \to \mathrm{Grpd}$ amounts to a diagram of groupoids $X \xrightarrow{f} Z \xleftarrow{g} Y$. The homotopy limit corresponding to Γ is called the *homotopy pullback* and is denoted $X \times^h_Z Y$. For objects, we have

$$X \times^h_Z Y = \{(x,y,z,\alpha,\beta) \mid x \in X, y \in Y, z \in Z, \alpha \in \mathrm{Hom}(f(x),z), \beta \in \mathrm{Hom}(g(y),z)\}.$$

Maps are given by the evident commutative diagrams. There is a canonical weak equivalence

$$X \times^h_Z Y \simeq \{(x,y,\phi) \mid x \in X, y \in Y, \phi \in \mathrm{Hom}(f(x),g(y))\},$$

given by the map $(x,y,z,\alpha,\beta) \mapsto (x,y,\beta^{-1}\alpha)$.

The homotopy pullback is homotopy invariant in the following sense: given a commutative diagram of groupoids

$$\begin{array}{ccccc} X & \longrightarrow & Z & \longleftarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ X_1 & \longrightarrow & Z_1 & \longleftarrow & Y_1 \end{array}$$

if all of the vertical maps are weak equivalences, then the induced canonical map $X \times^h_Z Y \to X_1 \times^h_{Z_1} Y_1$ is also a weak equivalence. A direct proof may be found in [Stacks, Tag 02XA]. It is also a special case of Proposition 3.6.

Example 3.2. Let Γ be a group. We abuse notation and write Γ for the category $\mathbb{B}\Gamma$. A functor $\Gamma \to \mathrm{Grpd}$ amounts to a groupoid X equipped with a Γ-action. In other words, Γ acts on the sets of objects and morphisms of X, in a manner compatible with the structure maps of X as a category. Set

$$X^{h\Gamma} = \operatorname*{holim}_{\longleftarrow\,\Gamma} X.$$

Then $X^{h\Gamma}$ is the groupoid of *homotopy fixed points* of the Γ-action. An object of $X^{h\Gamma}$ amounts to a pair (x,ϕ), where x is an object of X and ϕ is a rule that assigns, to each $g \in \Gamma$, a morphism $\phi(g)\colon x \to gx$ in X, satisfying $\phi(1) = \mathrm{id}$ and

$$\phi(gh) = g\phi(h) \circ \phi(g), \tag{3.2.1}$$

for all $g, h \in \Gamma$. An arrow $(x,\phi) \to (x_1,\phi_1)$ is a morphism $\alpha\colon x \to x_1$ in X such that

$$\phi_1(g) \circ \alpha = g\alpha \circ \phi(g), \tag{3.2.2}$$

for all $g \in \Gamma$. If $f : X \to Y$ is an equivariant map between groupoids equipped with Γ-actions, then we obtain a morphism of groupoids $f_*\colon X^{h\Gamma} \to Y^{h\Gamma}$ via

$$f_*((x,\phi)) = (f(x), g \mapsto f(\phi(g))), \qquad f_*(\alpha) = f(\alpha).$$

Example 3.3. The following special case of the above needs to singled out. Let $\Gamma = \mathbf{Z}$ and let $X \in \mathrm{Grpd}$. Consider the trivial action of $\mathbf{Z}$ on X. Then the homotopy fixed points $X^{h\mathbf{Z}}$ is the *free loop groupoid* of X, denoted LX. Its objects amount to

$$LX = \{(x, \phi) \mid x \in X, \phi \in \mathrm{Aut}(x)\}.$$

There is a canonical weak equivalence

$$LX \xrightarrow{\sim} X \times^h_{X \times X} X,$$

where the homotopy pullback is over the diagonal map $X \to X \times X$ (see Example 3.1 or [Stacks, Tag 04Z2]). The free loop groupoid is often called the *inertia groupoid* and denoted IX or ΛX in the literature.

The ordinary Γ-fixed points of X are not invariant under weak equivalence: the canonical map $\mathbb{E}\Gamma \to *$ is an acyclic fibration, but the Γ-action on $\mathbb{E}\Gamma$ is free (notation as in Example 2.1). Homotopy fixed points remedy this. The following is a special case of Proposition 3.6, but we provide an elementary independent proof.

Proposition 3.4. *If f is a weak equivalence, then so is f_*.*

Proof. We argue that the functor f_* is full, faithful and essentially surjective. Start with faith. It suffices to show that f_* is injective on automorphism groups. This is immediate from the definitions.

Now we show f_* is full. Let (x, ϕ) and (x_1, ϕ_1) be in $X^{h\Gamma}$, and let $\beta\colon f(x) \to f(x_1)$ be a morphism in Y corresponding to an arrow $(f(x), f(\phi)) \to (f(x_1), f(\phi_1))$ in Y_*. As f is full, there exists $\alpha\colon x \to x_1$ such that $f(\alpha) = \beta$. Further,

$$\begin{aligned}
f(\phi_1(g) \circ \alpha) &= f(\phi_1(g)) \circ \beta && \text{(as } f(\alpha) = \beta\text{)}\\
&= g\beta \circ f(\phi(g)) && \text{(by (3.2.2))}\\
&= gf(\alpha) \circ f(\phi(g)) && \text{(as } f(\alpha) = \beta\text{)}\\
&= f(g\alpha \circ \phi(g)) && \text{(since } f \text{ is equivariant).}
\end{aligned}$$

As f is faithful, this implies $\phi_1(g) \circ \alpha = g\alpha \circ \phi(g)$, for all $g \in \Gamma$. That is, α defines a map $(x, \phi) \to (x_1, \phi_1)$ in $X^{h\Gamma}$. By construction, this arrow is mapped by f_* to the morphism $(f(x), f(\phi)) \to (f(x_1), f(\phi_1))$ that we started with. Hence, f_* is full.

To show f_* is essentially surjective, let $(y, \psi) \in Y^{h\Gamma}$. As f is essentially surjective, there exists an isomorphism $\alpha\colon f(x) \to y$, for some $x \in X$. For $g \in \Gamma$, set

$$\gamma(g) = g\alpha^{-1} \circ \psi(g) \circ \alpha.$$

Then $\gamma(1) = \mathrm{id}$ and

$$\begin{aligned}
\gamma(gh) &= gh\alpha^{-1} \circ \psi(gh) \circ \alpha && \text{(by definition of } \gamma\text{)}\\
&= gh\alpha^{-1} \circ g\psi(h) \circ \psi(g) \circ \alpha && \text{(by (3.2.1))}\\
&= g(h\alpha^{-1} \circ \psi(h) \circ \alpha) \circ (g\alpha^{-1} \circ \psi(g) \circ \alpha) && \text{(as } g\alpha \circ g\alpha^{-1} = g(\mathrm{id}) = \mathrm{id}\text{)}\\
&= g\gamma(h) \circ \gamma(g) && \text{(by definition of } \gamma\text{),}
\end{aligned}$$

for all $g, h \in \Gamma$. Hence, $(f(x), \gamma) \in Y^{h\Gamma}$. As $\gamma(g)$ is a morphism $f(x) \to f(gx)$ and f is full, there exists $\phi(g)\colon x \to gx$ such that $f(\phi(g)) = \gamma(g)$ for every $g \in \Gamma$. As f is faithful, we must have $\phi(gh) = g\phi(h)\circ\phi(g)$. Thus, we obtain an object $(x,\phi)\in X^{h\Gamma}$ that is mapped by f to $(f(x),\gamma)$. Further, $\psi(g)\circ\alpha = g\alpha\circ\gamma(g)$. So α defines an isomorphism $(f(x),\gamma)\xrightarrow{\cong}(y,\psi)$. In other words, f_* is essentially surjective. □

Example 3.5. Let Γ be a finite group and let G be an arbitrary group on which Γ acts via automorphisms. Let $Z^1(\Gamma;G)$ be the set of G-valued 1-cocycles of Γ. I.e., functions $\sigma:\Gamma\to G$ satisfying

$$\sigma(gh) = \sigma(g)\cdot g\sigma(h)$$

for all $g,h\in\Gamma$. Let $\mathbb{E}_G Z^1(\Gamma;G)$ be the groupoid with set of objects $Z^1(\Gamma;G)$, and an arrow $\sigma\to\sigma_1$ for each $\alpha\in G$ such that

$$\sigma_1(g) = \alpha\cdot\sigma(g)\cdot g\alpha^{-1},$$

for all $g\in\Gamma$ (this is compatible with the notation of Example 2.1). The set of isomorphism classes of objects in $\mathbb{E}_G Z^1(\Gamma;G)$ is the first (non-abelian) cohomology $H^1(\Gamma;G)$ (see [S, Chapter 1.5]). The automorphism group of $\sigma\in\mathbb{E}_G Z^1(\Gamma;G)$ is

$$K_\sigma = \{\alpha\in G \mid \sigma(g)\cdot g\alpha\cdot\sigma(g)^{-1} = \alpha \text{ for all } g\in\Gamma\}.$$

Thus, there is a weak equivalence

$$\mathbb{E}_G Z^1(\Gamma;G)\simeq \coprod_{[\sigma]\in H^1(\Gamma;G)} \mathbb{B}K_\sigma,$$

This equivalence depends on picking an object in each isomorphism class $[\sigma]$. In particular, there is usually no way to make it canonical.

The Γ-action on G yields a Γ-action on $\mathbb{B}G$. Objects in $(\mathbb{B}G)^{h\Gamma}$ amount to functions $\phi:\Gamma\to G$ satisfying $\phi(1)=1$ and (3.2.1). To such a ϕ, assign a cocycle σ_ϕ by setting $\sigma_\phi(g) = \phi(g)^{-1}$, for all $g\in\Gamma$. This yields a canonical isomorphism

$$(\mathbb{B}G)^{h\Gamma}\xrightarrow{\cong}\mathbb{E}_G Z^1(\Gamma;G).$$

Hence, we obtain a weak equivalence

$$(\mathbb{B}G)^{h\Gamma}\simeq \coprod_{[\sigma]\in H^1(\Gamma;G)} \mathbb{B}K_\sigma.$$

In principle, this gives a presentation of $X^{h\Gamma}$ for an arbitrary groupoid X, since any groupoid is equivalent to one of the form $\coprod_i \mathbb{B}G_i$. As before, this involves picking objects in isomorphism classes and usually cannot be made canonical.

Proposition 3.6. *Let Γ be a small category and let $X:\alpha\mapsto X_\alpha$ and $Y:\alpha\mapsto Y_\alpha$ be Γ-diagrams in* Grpd*. Let $f:X\to Y$ be a natural transformation. If each map of groupoids $f_\alpha: X_\alpha\to Y_\alpha$ is a weak equivalence (resp. fibration), then $f_*:\underleftarrow{\operatorname{holim}}_\Gamma X\to\underleftarrow{\operatorname{holim}}_\Gamma Y$ is a weak equivalence (resp. fibration). Colloquially, homotopy limits preserve weak equivalences and fibrations.*

Proof. This is essentially [BK, Chapter XI 5.5-5.6]. However, [BK] is written in the language of simplicial sets. We orient the reader using the reference [Hir] since our notation is closer to the latter. Let B denote the nerve functor from the category of small categories to the category of simplicial sets. Then B is full, faithful and commutes with small limits. Consequently, if F and G are small categories, we have a canonical isomorphism [G, Section 4.4] ([T, Section 3] has a nice exposition):

$$B\mathrm{Map}(F,G) \xrightarrow{\cong} \mathrm{Map}(BF,BG),$$

where the right hand side is the simplicial function complex. Thus, $B\,\underleftarrow{\mathrm{holim}}_\Gamma X$ is canonically isomorphic to the equalizer of

$$\prod_{\alpha\in\Gamma} \mathrm{Map}(B(\Gamma\downarrow\alpha), BX_\alpha) \underset{B\psi}{\overset{B\phi}{\rightrightarrows}} \prod_{\{\sigma\colon \alpha\to\beta\}\in\Gamma} \mathrm{Map}(B(\Gamma\downarrow\alpha), BX_\beta)$$

and similarly for $B\,\underleftarrow{\mathrm{holim}}_\Gamma Y$. This is precisely the homotopy limit formula of [Hir, Definition 18.1.8]. As the nerve functor preserves and reflects weak equivalences as well as fibrations (see [A, Section 5]), we are reduced to showing that

$$Bf_*\colon \underleftarrow{\mathrm{holim}}_{\Gamma} BX \to \underleftarrow{\mathrm{holim}}_{\Gamma} BY$$

is a weak equivalence/fibration of simplicial sets in our situation. As each X_α and each Y_α is a groupoid, it follows that each BX_α and each BY_α is a Kan complex. In other words, BX and BY are diagrams in fibrant simplicial sets. Now the desired statement regarding weak equivalences is [Hir, Theorem 18.5.3 (2)], and that for fibrations is [Hir, Theorem 18.5.1 (2)]. □

Proposition 3.7. *Let Γ and Δ be small categories and let $X\colon (\gamma,\delta)\mapsto X_{(\gamma,\delta)}$ be a $\Gamma\times\Delta$-diagram in* Grpd. *Then we have canonical isomorphisms*

$$\underleftarrow{\mathrm{holim}}_{\Gamma}\,\underleftarrow{\mathrm{holim}}_{\Delta} X_{(\gamma,\delta)} \cong \underleftarrow{\mathrm{holim}}_{\Gamma\times\Delta} X \cong \underleftarrow{\mathrm{holim}}_{\Delta}\,\underleftarrow{\mathrm{holim}}_{\Gamma} X_{(\gamma,\delta)}.$$

Colloquially, homotopy limits commute with each other.

Proof. Use the nerve functor as in the proof of Proposition 3.6 and invoke [BK, Chapter XI 4.3]. Alternatively, note that for a small category K, the functor $\mathrm{Map}(K,-)$ commutes with limits and satisfies the exponential law. Further, limits always commute with each other. These statements applied to our formula for the homotopy limit give the desired result. □

4. **Filtered colimits of groupoids.** Like limits, all small colimits exist in the category of groupoids (see [Hig, Chapter 9] or [Hol, Appendix A]). In general, these can be nebulous to describe explicitly. However, filtered colimits present no such difficulties: they are given by filtered colimits of sets on objects and morphisms.

Lemma 4.1. *Filtered colimits of groupoids commute with finite products of groupoids.*

Proof. Immediate from the same for the category of sets. □

Lemma 4.2. *Let I be a small filtered category and let $X\colon i\mapsto X_i$ be an I-diagram in* Grpd. *If K is a finite category (i.e., has finitely many objects and morphisms), then the canonical map*

$$\underrightarrow{\lim}_{I}\,\mathrm{Map}(K,X_i) \to \mathrm{Map}(K,\underrightarrow{\lim}_{I} X_i)$$

is an isomorphism.

Proof. The inverse is defined as follows. As K is finite, an object of $\mathrm{Map}(K, \varinjlim_I X_i)$ amounts to picking a finite number of objects, and a finite number of morphisms satisfying a finite number of relations in $\varinjlim_I X_i$. As I is filtered, we may assume that these objects, morphisms and relations come from some X_j. This yields an object of $\varinjlim_I \mathrm{Map}(K, X_i)$ that is independent of the choices made. Similarly for morphisms. □

The heart of the matter is that the nerve of a groupoid is 2-coskeletal. So in this particular situation, the filtered colimit $\varinjlim_I \mathrm{Map}(K, X_i)$ behaves as if the nerve of K has only finitely many non-degenerate simplices. One may prove a generalization of Lemma 4.2 for n-coskeletal simplicial sets. However, its naïve analogue for *arbitrary* simplicial sets is false.[2] Here is a standard counterexample (simplicial sets and topological spaces are conflated as the argument is homotopy theoretic).

Let Γ be a finite group and let $B\Gamma$ be its classifying space. Let $\mathrm{sk}_n(B\Gamma)$ denote the n-skeleton of $B\Gamma$. The identity map on $B\Gamma$ does not factor through any finite skeleton. Consequently, $\varinjlim_n \mathrm{Map}(B\Gamma, \mathrm{sk}_n(B\Gamma))$ and $\mathrm{Map}(B\Gamma, B\Gamma)$ cannot be homotopy equivalent. Here, $\mathrm{Map}(-,-)$ denotes the set of continuous maps equipped with the compact open topology.

Proposition 4.3. *Let I be a small filtered category and let Γ be a finite category. Let $X : (i,\alpha) \mapsto X_{(i,\alpha)}$ be a $I \times \Gamma$-diagram in* Grpd. *Then the canonical map*

$$\varinjlim_I \underset{\Gamma}{\underleftarrow{\mathrm{holim}}}\, X_{(i,\alpha)} \to \underset{\Gamma}{\underleftarrow{\mathrm{holim}}} \varinjlim_I X_{(i,\alpha)}$$

is an isomorphism.

Proof. Apply Lemma 4.1 and Lemma 4.2. □

5. **Stacks.** Let $\mathscr{C}$ be a Grothendieck site with enough points. By definition of a point, the stalk of a sheaf of sets at a point is a left exact functor to the category of sets. On the other hand, a stalk is defined as a certain (small) colimit of sets (see [Stacks, Tag 00Y3]). Now a small category I is filtered if and only if colimits into the category of sets. indexed by I, commute with finite limits [KS, Theorem 3.1.6]. It follows that stalks are given by filtered colimits. This is also visibly obvious for most commonly used sites (the étale site on varieties, open subsets of a sober topological space, etc.).

Set

$$\mathrm{PreGrpd}(\mathscr{C}) = \text{category of presheaves of groupoids on } \mathscr{C}.$$

Note: 'presheaf' means 'strict presheaf' (as opposed to lax). We will not distinguish between an object $X \in \mathscr{C}$ and the presheaf $\mathrm{Hom}_{\mathscr{C}}(-, X)$ it represents.

A stalk for an object of $\mathrm{PreGrpd}(\mathscr{C})$ is defined using the same formula as for presheaves of sets. Each stalk is a groupoid. A map $X \to Y$ in $\mathrm{PreGrpd}(\mathscr{C})$ is called a *local weak equivalence* (resp. *local fibration*) if it induces a weak equivalence (resp. fibration) of groupoids on all stalks. The map is a *sectionwise weak equivalence* if it

[2] This 'canonical mistake' seems to be common (it would be ironic if I have missed some subtlety and fallen victim to it). The confusion being that there is a notion of 'finite ∞-category' (essentially a finite simplicial set, i.e., one with finitely many *non-degenerate simplices*). If the indexing category I is a finite ∞-category, then the statement *is true*. However, dropping the '∞' is asking for trouble. A finite (ordinary) category need not be a finite ∞-category (for example, $\mathbb{B}\mathbf{Z}/2\mathbf{Z}$).

induces a weak equivalence of groupoids $X(U) \to Y(U)$ for all $U \in \mathscr{C}$. A sectionwise weak equivalence is always a local weak equivalence. However, the converse does not generally hold.

In general, 'local' will refer to a property required to hold on all stalks. 'Sectionwise' will refer to a property required to hold on all sections.

An object in PreGrpd($\mathscr{C}$) is a stack if it is a sheaf of groupoids that satisfies effective descent. A map between stacks is a local weak equivalence if and only if it is a sectionwise weak equivalence [J, Proposition 9.28]. In other words, local weak equivalence for stacks amounts to the usual notion of equivalence of stacks. Analogous to the classical process of sheafification, the forgetful functor from stacks to PreGrpd($\mathscr{C}$) admits a left adjoint called stack completion (or stackification). For $X \in \mathrm{PreGrpd}(\mathscr{C})$, set

$$[X] = \text{stack completion of } X.$$

The unit of adjunction yields a canonical map $X \to [X]$ which is always a local weak equivalence [J, Corollary 9.27] (or see [Stacks, Tag 02ZO]).

Example 5.1. A morphism between sheaves of sets (viewed as groupoids) is a local weak equivalence if and only if it is an isomorphism.

Example 5.2. If $\mathscr{C} = *$, then PreGrpd($\mathscr{C}$) is the category of groupoids. A local weak equivalence amounts to an equivalence of categories.

Example 5.3. Let $\mathscr{C}$ be the big étale site on complex varieties.[3] Let G be a linear algebraic group. Define $BG \in \mathrm{PreGrpd}(\mathscr{C})$ by taking $BG(U)$ to consist of a single object with automorphism group $G(U)$. Let $[BG]$ denote the classifying stack of principal G-bundles. The map $BG \to [BG]$ sending the single object in $BG(U)$ to the trivial bundle on U is a local weak equivalence (étale local triviality of principal bundles). However, it is not a sectionwise weak equivalence.

6. **Homotopy limits: stacks.** Let Γ be a small category and let X be a Γ-diagram in PreGrpd($\mathscr{C}$). Define the homotopy limit $\underleftarrow{\mathrm{holim}}_{\Gamma} X$ sectionwise, i.e.,

$$(\underset{\Gamma}{\underleftarrow{\mathrm{holim}}} X)(U) = \underset{\Gamma}{\underleftarrow{\mathrm{holim}}} X(U),$$

for all $U \in \mathscr{C}$.

Example 6.1. As in Example 3.1, if Γ is the pullback category, we write $X \times_Z^h Y$ for the corresponding homotopy limit and call it the *homotopy pullback*. If all the objects involved in the pullback diagram are stacks $\mathscr{X}, \mathscr{Y}, \mathscr{Z}$, then

$$\mathscr{X} \times_{\mathscr{Z}}^h \mathscr{Y} \simeq \text{(2-)fibre product of stacks.}$$

Example 6.2. If Γ is a group then, as in Example 3.2, we set $X^{h\Gamma} = \underleftarrow{\mathrm{holim}}_{\Gamma} X$ and call it the *homotopy fixed points* of the Γ-action. If $\mathscr{X}$ is a stack (equipped with a Γ-action), then the definition of $\mathscr{X}^{h\Gamma}$ amounts to that of fixed point stack in [R]:

$$\mathscr{X}^{h\Gamma} \simeq \Gamma\text{-fixed point stack.}$$

[3]'Complex variety' = 'separated scheme over Spec($\mathbf{C}$)'.

Example 6.3. As in Example 3.3, if $X \in \mathrm{PreGrpd}(\mathscr{C})$, we write LX for the homotopy fixed points of $\mathbf{Z}$ acting on X trivially. If $\mathscr{X}$ is a stack, this amounts to the inertia stack

$$L\mathscr{X} \simeq \text{the inertia stack of } \mathscr{X}.$$

Theorem 6.4. *Let Γ be a small category and let $X\colon \alpha \mapsto X_\alpha$ be a Γ-diagram in $\mathrm{PreGrpd}(\mathscr{C})$. If each X_α is a stack, then $\underleftarrow{\mathrm{holim}}_\Gamma X$ is also a stack.*

Proof. Use the injective model structure of [Hol] ([J, Proposition 9.19] is a convenient reference). Let $Y \xrightarrow{\sim} RY$ denote a canonical fibrant model for $Y \in \mathrm{PreGrpd}(\mathscr{C})$. In particular, $Y \xrightarrow{\sim} RY$ is a local weak equivalence and RY is a fibrant object. Every fibrant object in $\mathrm{PreGrpd}(\mathscr{C})$ is a stack [J, Remark 9.23, Proposition 9.28]. Consequently, if each X_α is a stack, then each $X_\alpha \to RX_\alpha$ is a sectionwise weak equivalence. So, by Proposition 3.6, $\underleftarrow{\mathrm{holim}}_\Gamma X \to \underleftarrow{\mathrm{holim}}_\Gamma RX$ is a sectionwise weak equivalence. On the other hand, homotopy limits preserve fibrant objects [Hir, Theorem 18.5.2 (2)]. In particular, $\underleftarrow{\mathrm{holim}}_\Gamma RX$ is fibrant and hence a stack. Thus, $\underleftarrow{\mathrm{holim}}_\Gamma X$ is a stack. □

Theorem 6.5. *Let Γ and Δ be small categories and let $X\colon (\gamma,\delta) \mapsto X_{(\gamma,\delta)}$ be a $\Gamma \times \Delta$-diagram in $\mathrm{PreGrpd}(\mathscr{C})$. Then we have canonical isomorphisms*

$$\underset{\Gamma}{\underleftarrow{\mathrm{holim}}}\,\underset{\Delta}{\underleftarrow{\mathrm{holim}}}\,X_{(\gamma,\delta)} \cong \underset{\Gamma\times\Delta}{\underleftarrow{\mathrm{holim}}}\,X \cong \underset{\Delta}{\underleftarrow{\mathrm{holim}}}\,\underset{\Gamma}{\underleftarrow{\mathrm{holim}}}\,X_{(\gamma,\delta)}.$$

Colloquially, homotopy limits commute with each other.

Proof. Immediate from Proposition 3.7, since we have defined homotopy limits in $\mathrm{PreGrpd}(\mathscr{C})$ by a sectionwise prescription. □

Corollary 6.6. *Let $\mathscr{X}, \mathscr{Y}, \mathscr{Z}$ be stacks equipped with the action of a group Γ. Suppose $\mathscr{X} \to \mathscr{Z}$ and $\mathscr{Y} \to \mathscr{Z}$ are Γ-equivariant maps. Then the stacks $(\mathscr{X} \times^h_{\mathscr{Z}} \mathscr{Y})^{h\Gamma}$ and $\mathscr{X}^{h\Gamma} \times^h_{\mathscr{Z}^{h\Gamma}} \mathscr{Y}^{h\Gamma}$ are canonically equivalent.*

Corollary 6.7. *Let $\mathscr{X}$ be a stack equipped with the action of a group Γ. Then the stacks $(L\mathscr{X})^{h\Gamma}$ and $L\mathscr{X}^{h\Gamma}$ are canonically equivalent.*

We now move to results that are specific to finite diagrams. These follow from the corresponding results for groupoids combined with Proposition 4.3.

Theorem 6.8. *Let Γ be a finite category and let $f\colon X \to Y$ be a natural transformation of Γ-diagrams in $\mathrm{PreGrpd}(\mathscr{C})$. If each $f_\alpha\colon X_\alpha \to Y_\alpha$ is a local weak equivalence (resp. local fibration), then the induced map*

$$f_*\colon \underset{\Gamma}{\underleftarrow{\mathrm{holim}}}\,X \to \underset{\Gamma}{\underleftarrow{\mathrm{holim}}}\,Y$$

is a local weak equivalence (resp. local fibration).

Proof. Apply Proposition 3.6 and Proposition 4.3. □

Theorem 6.9. *If Γ is a finite category, then the stacks $[\underleftarrow{\mathrm{holim}}_\Gamma X]$ and $\underleftarrow{\mathrm{holim}}_\Gamma [X]$ are canonically equivalent.*

Proof. The canonical map $X_\alpha \to [X_\alpha]$ is a local weak equivalence for each α. So, if Γ is a finite category, $\underleftarrow{\mathrm{holim}}_\Gamma X \to \underleftarrow{\mathrm{holim}}_\Gamma [X]$ is a local weak equivalence by Theorem

6.8. As $\underleftarrow{\operatorname{holim}}_{\Gamma}[X]$ is already a stack by Theorem 6.4, applying stack completion yields the desired result. □

Informally, Theorem 6.9 says that homotopy limits of *finite* diagrams of stacks may be computed correctly using presentations via groupoids. The utility here is that such presentations are often more amenable to computation than the stack itself. In analogy with homological algebra, stacks are like large injective resolutions - theoretically useful, but usually too large for any sort of explicit computation. For homotopy pullbacks and inertia stacks this is well known (see [Stacks, Tags 026G, 036X]). We formally record it for fixed point stacks.

Corollary 6.10. *Let Γ be a finite group acting on $X \in \mathrm{PreGrpd}$. Then the stacks $[X^{h\Gamma}]$ and $[X]^{h\Gamma}$ are canonically equivalent.*

Theorem 6.11. *Let I be a small filtered category and let Γ be a finite category. Let $X\colon (i,\alpha) \mapsto X_{(i,\alpha)}$ be a $I \times \Gamma$-diagram in $\mathrm{PreGrpd}(\mathscr{C})$. Then the canonical map*

$$\varinjlim_{I} \underleftarrow{\operatorname{holim}}_{\Gamma} X_{(i,\alpha)} \to \underleftarrow{\operatorname{holim}}_{\Gamma} \varinjlim_{I} X_{(i,\alpha)}$$

is an isomorphism.

Proof. Apply Proposition 4.3. □

Example 6.12. Let $\mathscr{C}$ be the big étale site of complex varieties. Let Γ be a finite group acting on a linear algebraic group G via automorphisms. We will use the notation of Example 3.5 and Example 5.3. In particular, $BG(U)$ denotes the groupoid consisting of a single object with automorphisms $G(U)$, for all $U \in \mathscr{C}$; $[BG]$ is its stack completion - the stack of principal G-bundles; $Z^1(\Gamma; G(U))$ is the set of 1-cocycles, etc. Then Γ acts on BG, and we have a canonical isomorphism

$$(BG)^{h\Gamma} \xrightarrow{\cong} E_G Z^1(\Gamma; G),$$

where, for $U \in \mathscr{C}$, in the notation of Example 3.5,

$$E_G Z^1(\Gamma; G)(U) = \mathbb{E}_{G(U)} Z^1(\Gamma; G(U)).$$

As Γ is finite, and in particular has a finite presentation, the presheaf $U \mapsto Z^1(\Gamma; G(U))$ is representable by an affine G-variety Z. Explicitly, Z is the variety of group homomorphisms $\Gamma \to G \rtimes \Gamma$ that are sections of the canonical map $G \rtimes \Gamma \to \Gamma$.

Embed $G \rtimes \Gamma$ in some GL_n and consider the representation variety Y of group homomorphisms $\Gamma \to GL_n$. Every representation of Γ over $\mathbf{C}$ is completely reducible. Further, for a fixed dimension there are finitely many isomorphism classes of these. So, under the conjugation action of GL_n, the variety Y has finitely many orbits and each orbit is closed [PR, Theorem 2.17]. On the other hand, Z embeds into Y, and each G-orbit in Z is the intersection of some GL_n-orbit with Z (see [PR, Lemma 2.11 and Theorem 2.17]). Thus, there are finitely many G-orbits in Z and they are all closed. Consequently, Z is isomorphic to the disjoint union of its orbits. These orbits are parametrized by $H^1(\Gamma; G(\mathbf{C}))$ (see Example 3.5). Moreover, for a geometric point $\sigma \in Z^1(\Gamma; G(\mathbf{C}))$, the stabilizer of σ is

$$K_\sigma = \{\alpha \in G \mid \sigma(g) \cdot g\alpha \cdot \sigma(g)^{-1} = \alpha \text{ for all } g \in \Gamma\}.$$

Thus, we have a local weak equivalence

$$(BG)^{h\Gamma} \simeq \coprod_{[\sigma]\in H^1(\Gamma;G(\mathbf{C}))} BK_\sigma.$$

As in Example 3.5, it is generally impossible to make this canonical.

Write $[Z/G]$ for the stack completion of $E_G Z^1(\Gamma;G)$, i.e., the stack quotient of Z by G. By Corollary 6.10, we have equivalences of stacks:

$$[BG]^{h\Gamma} \simeq [Z/G] \simeq \coprod_{[\sigma]\in H^1(\Gamma;G(\mathbf{C}))} [BK_\sigma].$$

The first of these equivalences is canonical. However, the second depends on the various choices made above and cannot usually be made canonical. In particular, it is generally impossible to identify the first and last spaces here. Regardless, it has computational utility. Readers may explore the consequences of Corollary 6.6 and Corollary 6.7 in this setting at their leisure.

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The Appalachians