COHOMOLOGY OF HOMOGENEOUS VARIETIES IS TATE

R. VIRK

The purpose of this note is to make the following observation:

Proposition. *Let G be a linear algebraic group and $H \subset G$ a closed subgroup. Then the cohomology $H^*(G/H)$ is Tate.*

The Tate property refers to the mixed Hodge structure on the cohomology of a variety (see [D]) - a (mixed) Hodge structure is called Tate if it is a successive extension of Hodge structures of type (n, n) (the n is allowed to vary). A variety means a separated scheme of finite type over the complex numbers $\mathbf{C}$. Cohomology is taken with rational coefficients and with respect to the complex analytic site.

The Proposition follows from the following well known result combined with the result immediately after.

Lemma. *Let G be a linear algebraic group. Then $H^*(G)$ is Tate.*

Proof. We may assume G is connected reductive (Levi decomposition). Let $T \subset G$ be a maximal torus. Then $H^*(G) \to H^*(T)$ is injective (splitting principle). As $H^*(T)$ is clearly Tate, we are done. □

Theorem. *Let G be a linear algebraic group acting on a variety X. If $H^*(X)$ is Tate, then the G-equivariant cohomology $H^*_G(X)$ is Tate.*

Details on endowing equivariant cohomology with functorial Hodge structures can be found in [D].

Proof. Consider the category whose objects are points $x \in X(\mathbf{C})$ and morphisms consist of $x \to y$ for each $g \in G(\mathbf{C})$ such that $gx = y$. Let X_{hG} be the nerve of this category. Then X_{hG} is a semi-simplicial variety, and $H^*_G(X) = H^*(X_{hG})$. The degree q piece of X_{hG} is $G^{\times q} \times X$. Consequently, we obtain a spectral sequence, converging to $H^*(X_{hG})$, with terms $H^p(G^{\times q} \times X)$. Both $H^*(G)$ and $H^*(X)$ are Tate. Thus, applying the Künneth formula, $H^*(G^{\times q} \times X)$ is Tate. Consequently, $H^*(X_{hG})$ is Tate. □

Similar statements, with exactly the same proofs, can also be made for varieties over finite fields and ℓ-adic cohomology. Both the Hodge and the ℓ-adic versions are realizations of a general motivic result which is discussed in [SVW].[1]

References

[D] P. Deligne, *Théorie de Hodge III*, Pub I.H.É.S **44** (1974), 5-77.
[SVW] W. Soergel, M. Wendt, R. Virk, *Equivariant motives and representation theory*, in preparation.

The Appalachians

[1] This note is extracted from much more general arguments in [SVW]. However, as homogeneous spaces abound in nature, and the basic simplicial argument (essentially a disguised and adapted version of the Eilenberg-Moore spectral sequence) is so simple, maybe the present standalone writeup will be of some interest.