KÜNNETH STUFF

R. VIRK

1. All spaces will be locally compact. I might want some more conditions like finite cohomological dimension, etc., but let me ignore this for now and just assume that all the usual hygiene regulations are obeyed.

2. Let

$$\begin{array}{ccc} E & \xrightarrow{g} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{q} & S \end{array}$$

be a cartesian square. Set

$$a = p \circ g = q \circ f.$$

Proposition 2.1. *Let M and N be objects (not neccesarily constructible) on X and Y respectively. Then we have a canonical isomorphism*

$$a_!(g^*M \otimes f^*N) \simeq p_!M \otimes q_!N.$$

Proof. We have

$$\begin{aligned} a_!(g^*M \otimes f^*N) &\simeq q_!f_!(g^*M \otimes f^*N) \\ &\simeq q_!(f_!g^*M \otimes N) \qquad \text{(projection formula)} \\ &\simeq q_!(q^*p_!M \otimes N) \qquad \text{(proper base change)} \\ &\simeq p_!M \otimes q_!N \qquad \text{(projection formula).} \end{aligned}$$

□

3. From now on we assume that

$$S = \mathrm{pt} \quad \text{so that} \quad E = X \times Y.$$

Proposition 3.1. *Let M and N be constructible. Then*

$$\mathbf{D}(M \boxtimes N) \simeq \mathbf{D}M \boxtimes \mathbf{D}N.$$

Proof. According to Kashiwara-Schapira, "Sheaves on Manifolds" (Proposition 3.4.4), we have

$$\mathbf{D}M \boxtimes \mathbf{D}N \simeq \mathscr{H}om(g^*M, f^!\mathbf{D}N),$$

where $\mathscr{H}om$ denotes inner Hom. Now if we let ω denote the dualizing object, then

$$\begin{aligned} \mathbf{D}(M \boxtimes N) &\simeq \mathscr{H}om(g^*M \otimes f^*N, \omega) \\ &\simeq \mathscr{H}om(g^*M, \mathscr{H}om(f^*N, \omega)) \\ &\simeq \mathscr{H}om(g^*M, \mathbf{D}f^*N) \\ &\simeq \mathscr{H}om(g^*M, f^!\mathbf{D}N). \end{aligned}$$

□

Proposition 3.2. *Assume that over a point duality (anti)-commutes with tensor product (for instance if our sheaves are with field coefficients). Let M and N be constructible. Then*

$$a_*(M \boxtimes N) = q_* N \otimes p_* M.$$

Proof. Apply $\mathbf{D}$ to the isomorphism

$$a_!(\mathbf{D}M \boxtimes \mathbf{D}N) = p_! \mathbf{D}M \otimes q_! \mathbf{D}N$$

to get

$$a_*(M \boxtimes N) = \mathbf{D}(p_! \mathbf{D}M \otimes q_! \mathbf{D}N).$$

Now use the assumption on duality commuting with tensor product. □

The Appalachians