The Appalachians

October 4, 2014

Dear Mark,

I will deduce all the statements about Grothendieck groups, finite length, Jordan-Hölder series, etc. from the following result. I am postponing the proof to the end so as to keep the discussion technicality free in the beginning.

Proposition. *Let $f\colon X \to Y$ be a G-equivariant morphism between G-varieties. If f is smooth of relative dimension d with all fibres connected, then*

$$f^*[d]\colon \mathrm{Perv}_G(Y) \to \mathrm{Perv}_G(X)$$

is full and faithful. Moreover, the image of $\mathrm{Perv}_G(Y)$ in $\mathrm{Perv}_G(X)$ is closed under taking subquotients.

The stability under taking subquotients immediately yields:

Corollary. *Let f be as above. If $\mathcal{L} \in \mathrm{Perv}_G(Y)$ is irreducible, then so is $f^*\mathcal{L}[d]$.*

Corollary. *Each $\mathcal{A} \in \mathrm{Perv}_G(Y)$ has finite length and satisfies the Jordan-Hölder property.*

Proof. We may find (using Steifel varieties for instance) a free G-space X along with a smooth morphism $f\colon X \to Y$ of relative dimension d with connected fibres. By our Proposition and the preceding Corollary it suffices to demonstrate the assertion for $f^*\mathcal{A}[d]$. As X is a free G-space, we have $D_G(X) \simeq D(G\backslash X)$. So our assertion reduces to the corresponding statement in the non-equivariant context. **Q.E.D.**

As we previously discussed, the finite length and Jordan-Hölder property yield that $K_0(D_G(Y))$ is free abelian (with the classes of irreducibles giving a basis). As you also pointed out, putting all of the above together we obtain:

Corollary. *Let f be as above. Then the induced map*

$$f^*\colon K_0(D_G(Y)) \to K_0(D_G(X))$$

is injective.

We also have:

Corollary. *If G is connected, then the forgetful functor $\mathrm{Perv}_G(Y) \to \mathrm{Perv}(Y)$ is full and faithful. Moreover the image of $\mathrm{Perv}_G(Y)$ in $\mathrm{Perv}(Y)$ is closed under taking subquotients.*

Proof. Let G act on $G \times Y$ diagonally. Write $f\colon G \times Y \to Y$ for the projection. Then the forgetful functor is the composition

$$\mathrm{Perv}_G(Y) \xrightarrow{f^*[\dim G]} \mathrm{Perv}_G(G \times X) \xrightarrow{\sim} \mathrm{Perv}(X).$$

As G is connected, f satisfies the assumptions of the Proposition. **Q.E.D.**

Let me also give the traditional proof of the preceding Corollary. It looks a lot messier, but I maintain that at its heart the argument is the same.

Let $a\colon G \times Y \to Y$ be the action map, and let $p\colon G \times Y \to Y$ be the projection. Let $e\colon \mathrm{pt} \to G$ be the identity section, and $m\colon G \times G \to G$ the multiplication. Then $\mathrm{Perv}_G(Y)$ is equivalent to the category of pairs $(\mathcal{A}, \sigma)$ with $\mathcal{A} \in \mathrm{Perv}(Y)$ and $\sigma\colon a^*\mathcal{A} \xrightarrow{\sim} p^*\mathcal{A}$ an isomorphism satisfying:

(i) $(e \times \mathrm{id})^*(\sigma) = \mathrm{id}_{\mathcal{A}}$;

(ii) $(m \times \mathrm{id})^*(\sigma) = (\mathrm{id} \times p)^*(\sigma) \circ (\mathrm{id} \times a)^*(\sigma)$.

The forgetful functor $\mathrm{Perv}_G(Y) \to \mathrm{Perv}(Y)$ is given by $(\mathcal{A}, \sigma) \mapsto \mathcal{A}$. Let me comment that one obtains an equivalent category if one omits (i). This is because given a $\sigma\colon a^*\mathcal{A} \xrightarrow{\sim} p^*\mathcal{A}$, we can replace it with $\sigma \circ a^*(e \times \mathrm{id})^*(\sigma^{-1})$ to obtain an isomorphic object satisfying (i).

Let me remind you that the non-equivariant analogue of our key Proposition is known ([*Faisceaux Pervers, Proposition 4.2.5 and Corollary 4.2.6.2*]). Namely:

Proposition (non-equivariant version). *Let $f\colon X \to Y$ be a smooth morphism of varieties, of relative dimension d. If the fibres of f are connected, then*

$$f^*[d]\colon \mathrm{Perv}(Y) \to \mathrm{Perv}(X)$$

is full and faithful. Moreover, the image of $\mathrm{Perv}(Y)$ *in* $\mathrm{Perv}(X)$ *is closed under taking subquotients.*

Observation. *With f as above, assume we are given a map $s\colon Y \to X$ which is a section of f. Then $s^*[-d]$ restricted to the essential image of $f^*[d]\colon$* $\mathrm{Perv}(Y) \to \mathrm{Perv}(X)$ *gives the inverse functor to $f^*[d]$.*

Proof. Immediate from $s^* f^* = \mathrm{id}$. **Q.E.D.**

With all of this in hand one can now give the traditional proof for the forgetful functor being full and faithful. More precisely:

Scholium. *Assume G is connected. Then the forgetful functor* $\mathrm{Perv}_G(Y) \to \mathrm{Perv}(Y)$ *is full and faithful. Its image is closed under taking subquotients. Moreover this image consists of those $\mathcal{A} \in$* $\mathrm{Perv}(Y)$ *for which there merely exists an isomorphism $a^*\mathcal{A} \simeq p^*\mathcal{A}$.*

Proof. Let me start by identifying the essential image. We need to show that for $\mathcal{A} \in \mathrm{Perv}(Y)$ if an isomorphism $\sigma\colon a^*\mathcal{A} \xrightarrow{\sim} p^*\mathcal{A}$ exists, then one can upgrade it to an isomorphism satisfying the compatibilities (i) and (ii). As commented earlier we can assume σ satisfies (i). Then I claim that σ automatically satisfies (ii). Indeed, (ii) asks for the equality of two maps

$$(m \times \mathrm{id})^* a^* \mathcal{A} \to (\mathrm{id} \times p)^* p^* \mathcal{A}.$$

Both of these maps give $\mathrm{id}_{\mathcal{A}}$ under $(e \times \mathrm{id})^*(e \times \mathrm{id} \times \mathrm{id})^*$. Consequently, applying our Observation regarding sections yields that the maps must be equal to start with.

Now let's show full faithfulness. Suppose $(\mathcal{A}_1, \sigma_1)$ and $(\mathcal{A}_2, \sigma_2)$ are in $\mathrm{Perv}_G(Y)$. We need to show that any map $\phi\colon \mathcal{A}_1 \to \mathcal{A}_2$ automatically intertwines σ_1 and σ_2. But this is again asking for the equality of two maps

$$a^*\mathcal{A}_1 \to a^*\mathcal{A}_2.$$

Both of these maps give ϕ under $(e \times \mathrm{id})^*$ (by compatibility (i)). So applying our Observation regarding sections once again yields the desired equality.

Finally, let me show the stability under subquotients. Let $\mathcal{A} \in \mathrm{Perv}(Y)$ with $a^*\mathcal{A} \simeq p^*\mathcal{A}$, and let $\mathcal{A}_1$ be a subquotient of $\mathcal{A}$. Then by the Proposition (non-equivariant version) there is some $\mathcal{A}_2 \in \mathrm{Perv}(Y)$ with $p^*\mathcal{A}_2 \simeq a^*\mathcal{A}_1$. Applying e^* to this isomorphism yields $\mathcal{A}_2 \simeq \mathcal{A}_1$. Consequently, $a^*\mathcal{A}_1 \simeq p^*\mathcal{A}_2 \simeq p^*\mathcal{A}_1$. **Q.E.D.**

As an aside let me mention that one could question why we only ask for the compatibilities (i) and (ii) for equivariant perverse sheaves? These are compatibilities from the first part of the simplicial object $[G\backslash X]$. Shouldn't one ask for compatibilities corresponding to the full object $[G\backslash X]$? The compatibilities (i) and (ii) are actually enough to guarantee all the remaining ones. This is proved by using the same sort of argument as above using our Observation on sections (perverse sheaves form a stack; surjective smooth morphisms have étale local sections). I will confess though that I have never checked the details of this.

Ok, I think it's time to prove the key Proposition stated at the beginning. I need to start by recalling some facts about equivariant derived categories.

The equivariant derived category $D_G(Y)$ comes equipped with a forgetful functor $\mathrm{For}\colon D_G(Y) \to D(Y)$ to the ordinary derived category $D(Y)$. The functor For is t-exact, conservative and commutes with pullbacks.

For any integer a denote by $D^{\leq a}(Y)$ the full subcategory consisting of objects $\mathcal{A} \in D(Y)$ which satisfy $H^i(Y) = 0$ for $i > a$. Define $D^{\geq a}(X)$ similarly. Given a segment $I = [a, b] \subseteq \mathbf{Z}$, let

$$D^I(Y) = D^{\geq a}(X) \cap D^{\leq b}(Y).$$

Further, set $|I| = b - a$. Finally, define $D^I_G(Y)$ using the forgetful functor. That is, $\mathcal{A} \in D^I_G(Y)$ if $\mathrm{For}(A) \in D^I(Y)$.

If G acts freely on Y, then there is an equivalence $D_G(Y) \simeq D(G\backslash Y)$. This equivalence commutes with pullbacks.

Let me also be a bit precise about the meaning of n-acyclic. A map $f\colon X \to Y$ is called *n-acyclic* if:

(i) for any sheaf $\mathcal{F} \in \mathrm{Sh}(Y) \subseteq D(X)$ the canonical map

$$\mathcal{F} \to \tau_{\leq n} f_* f^* \mathcal{F}$$

is an isomorphism; here $\tau_{\leq n}$ is the truncation corresponding to the standard t-structure;

(ii) each base change of f also satisfies the above property.

In the literature such maps are sometimes called universally n-acyclic, presumably to emphasize the base change property. A topological criterion for such maps is as you would expect: a locally trivial fibration with connected fibres whose cohomology vanishes in the interval $(0, n]$.

Suppose we are given an n-acyclic G-equivariant map $f\colon X \to Y$ with G acting on X freely. Let $q\colon X \to G\backslash X$ be the quotient map.

$$Y \xleftarrow{f} X \xrightarrow{q} G\backslash X$$

Then for any segment I with $|I| \leq n$ the category $D^I_G(Y)$ is equivalent to the full subcategory of $D^I(G\backslash X)$ consisting of $\overline{\mathcal{A}} \in D^I(G\backslash X)$ such that there exists an isomorphism $f^*\mathcal{A}_Y \simeq q^*\overline{\mathcal{A}}$ for some $\mathcal{A}_Y \in D^I(Y)$ (pullbacks on both sides here are the ordinary/non-equivariant pullbacks). This is in some sense the *definition* of $D_G(Y)$.

Observe that if one takes $Y = X$ and $f = \mathrm{id}$, then this says that $D^I_G(X)$ is equivalent to $D^I(G\backslash X)$ for all I. This is just the equivalence $D_G(X) \simeq D(G\backslash X)$. Furthermore, in the context of these descriptions, the equivariant pullback $f^*\colon D^I_G(Y) \to D^I_G(X)$ is given by mapping $\overline{\mathcal{A}}$ to itself. In particular, we tautologically have that $f^*\colon D^I_G(Y) \to D^I_G(X)$ is full and faithful.

We can now prove the following special case of the key Proposition.

Lemma. *Let $f\colon X \to Y$ be a G-equivariant smooth morphism of relative dimension d with connected fibres. Assume that G acts on X freely and that f is* $\dim Y$*-acyclic.*

Then

$$f^*[d] \colon \mathrm{Perv}_G(Y) \to \mathrm{Perv}_G(X)$$

is full and faithful. Moreover, the essential image of $\mathrm{Perv}_G(Y)$ *in* $\mathrm{Perv}_G(X)$ *is closed under taking subquotients.*

Proof. Note that $\mathrm{Perv}_G(Y) \subseteq D_G^{[-\dim Y,0]}(Y)$ so the full and faithfulness has already been observed.

To demonstrate the stability under subquotients I will use the notation and descriptions from the discussion above. Further, set $d_G = \dim G$.

We need to show that if $\overline{\mathcal{A}} \in \mathrm{Perv}(G\backslash X)$ is such that $q^*\overline{\mathcal{A}}[d_G]$ is isomorphic to $f^*\mathcal{A}_Y[d]$ for some $\mathcal{A}_Y \in \mathrm{Perv}(Y)$, and $\overline{\mathcal{A}}'$ is a subquotient of $\overline{\mathcal{A}}$, then $q^*\overline{\mathcal{A}}'[d_G]$ is isomorphic to $f^*\mathcal{A}'_Y[d]$ for some $\mathcal{A}'_Y \in \mathrm{Perv}(Y)$.

Now $q^*\overline{\mathcal{A}}'[d_G]$ is a subquotient of $q^*\overline{\mathcal{A}}[d_G]$. As $q^*\overline{\mathcal{A}}[d_G]$ is in the image of $f^*[d] \colon \mathrm{Perv}(Y) \to \mathrm{Perv}(X)$, the Proposition (non-equivariant version) yields the desired $\mathcal{A}'_Y$. **Q.E.D.**

Great! The proof of the key Proposition is now straightforward. To save you the bother of flipping back, let me remind you that we want to prove:

Proposition. *Let $f \colon X \to Y$ be a G-equivariant morphism between G-varieties. If f is smooth of relative dimension d with all fibres connected, then*

$$f^*[d] \colon \mathrm{Perv}_G(Y) \to \mathrm{Perv}_G(X)$$

is full and faithful. Moreover, the image of $\mathrm{Perv}_G(Y)$ *in* $\mathrm{Perv}_G(X)$ *is closed under taking subquotients.*

Proof. If G acts freely on Y, then the claim immediately reduces to its non-equivariant analogue. The general case reduces to the case of G acting freely as follows. Let $n = \dim Y$. Note that $\mathrm{Perv}_G(Y) \subseteq D_G^{[-n,0]}(X)$. Pick any smooth n-acyclic map $p \colon Z \to Y$. Let $\widetilde{Z} = Z \times_Y Z$ so that we have a Cartesian square:

$$\begin{array}{ccc} \widetilde{Z} & \xrightarrow{\tilde{f}} & Z \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & Y \end{array}$$

By the preceding Lemma, $p^*[n]$, $\tilde{p}^*[n]$ are full and faithful on the corresponding categories of equivariant perverse sheaves. Further, their essential images are closed under taking subquotients. Consequently, it suffices to show that $\tilde{f}^*[d] \colon \mathrm{Perv}_G(Z) \to \mathrm{Perv}_G(\widetilde{Z})$ is full, faithful and its image is closed under taking subquotients. This reduces us to the case of free actions and we are done. **Q.E.D.**

Ok, rigor aside, let me make some informal remarks to explain why any general result in the non-equivariant setting will almost always carry over to the equivariant setting.

If one takes the Borel picture of equivariant cohomology to heart, then $D_G(X)$ really should be a full subcategory of $D(EG \times^G X)$. A G-equivariant morphism $X \to Y$ will induce a morphism $D(EG \times^G X) \to D(EG \times^G Y)$. This allows one to define functors on $D_G(X)$ by using the corresponding functors on $D(EG \times^G X)$. Moreover, any reasonable property of the morphism $X \to Y$ will carry over to one of $EG \times^G X \to EG \times^G Y$. Thus, any properties that our functors or categories satisfy in the non-equivariant setting will carry over to the equivariant context.

The only problem with all of this is that $EG \times^G X$ is usually infinite dimensional in nature. The remedy is to capture pieces of $D(EG \times^G X)$ using approximations to $EG \times^G X$. This is formalized with the whole business involving acyclic maps, Steifel varieties and the $D_G^I(X)$.

Thus, to prove anything about $D_G(X)$ we want to first move to a large enough approximation that captures all the data we are interested in, and then to just use a known result in the non-equivariant context. This is exactly what is done in almost every single argument above. A sufficiently motivated person could probably even write a meta Theorem that could be used to bludgeon all of these sort of results in one swing.

On a related note it might amuse you to know that the original version of my preprint had something along the lines of 'Immediate from the non-equivariant version.' as proof for all the equivariant results. Then someone yelled at me, so I put in the details you see in the more recent version.

I hope that at least some of this helps!

Best,
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