PROJECTIVE FUNCTORS

Last updated: 20 June 2013

Apart from sins of omission, exposition, and failings in my understanding, there is nothing particularly original in this note. It stems from my attempts to understand [BG] in topological language.

0.1. **Notation.** For a variety X, write $D(X)$ for the bounded derived category of constructible sheaves on X. Let $\mathrm{Perv}(X) \subset D(X)$ be the abelian subcategory of perverse sheaves (middle perversity), and write ${}^pH^*$ for the cohomological functor corresponding to the perverse t-structure. The constant (rank 1) sheaf on X will be denoted by $\underline{X}$. If an algebraic group H acts on X, write $D(H\backslash X)$ and $\mathrm{Perv}(H\backslash X)$ for the corresponding equivariant categories. In this document the group H occuring in such a situation will always be connected. Consequently, we can and will identify $\mathrm{Perv}(H\backslash X)$ with a full subcategory of $\mathrm{Perv}(X)$.

0.2. **Convolution formalism.** Let G be an algebraic group, $H \subseteq G$ a closed subgroup, and X a variety with G-action. Assume that the geometric quotient G/H exists. Let $G \overset{H}{\times} X \to G/H$ denote the X-fibre bundle associated to the H-torsor $G \to G/H$. Let $m\colon G \overset{H}{\times} X \to X$ be the morphism induced by the G-action. For $M \in D(H\backslash G/H)$ and $N \in D(H\backslash X)$, let $M \mathbin{\widetilde{\boxtimes}} N$ denote the object in $D(H\backslash(G \overset{H}{\times} X))$ whose pullback to $G \times X$ coincides with the pullback of $M \boxtimes N$. The *convolution* bifunctor $- \cdot -\colon D(H\backslash G/H) \times D(H\backslash X) \to D(H\backslash X)$ is defined by

$$M \cdot N = m_!(M \mathbin{\widetilde{\boxtimes}} N).$$

Taking $X = G/H$ yields a monoidal structure on $D(H\backslash G/H)$. If m is proper, then convolution commutes with Verdier duality.

0.3. **Flag variety.** From now on $G^\vee$ will be a connected reductive linear algebraic group, $B^\vee \subseteq G^\vee$ a Borel subgroup, and $U^\vee \subseteq B^\vee$ the unipotent radical of $B^\vee$. Let $T^\vee = B^\vee/U^\vee$ be the (abstract) maximal torus. Write W for the Weyl group, and $\ell\colon W \to \mathbf{Z}_{\geq 0}$ for the length function.

The Bruhat decomposition yields

$$\mathscr{F}\ell = \bigsqcup_{w \in W} \mathscr{F}\ell_w, \text{ where } \mathscr{F}\ell_w = B^\vee w B^\vee/B^\vee, \text{ and } \mathscr{F}\ell_w \simeq \mathbf{C}^{\ell(w)}.$$

For each $w \in W$, put

$$\mathbf{T}_w = i_{w!}\underline{\mathscr{F}\ell_w},$$

where $i_w\colon \mathscr{F}\ell_w \hookrightarrow \mathscr{F}\ell$ is the inclusion. Convolution formalism yields a monoidal structure on $D(B^\vee\backslash\mathscr{F}\ell)$. The unit object is $\mathbf{1} = \mathbf{T}_e$. Using the Bruhat decomposition one infers that the $\mathbf{T}_w$ satisfy the braid relations:

$$\text{if } \ell(vw) = \ell(v) + \ell(w), \text{ then } \mathbf{T}_v \cdot \mathbf{T}_w = \mathbf{T}_{vw}.$$

Further, as each i_w is affine, the functor $\mathbf{T}_w \cdot -[\ell(w)]$ (resp. $\mathbf{DT}_w \cdot -[-\ell(w)]$) is left (resp. right) t-exact. It is straightforward to see that $\mathrm{supp}(\mathbf{T}_w \cdot \mathbf{DT}_{w^{-1}}) = \mathscr{F}\ell_e$. Consequently, each $\mathbf{T}_w$ is invertible, with inverse $\mathbf{T}_w^{-1} = \mathbf{DT}_{w^{-1}}$.

0.4. **Enhanced flag variety.** Let $\widetilde{\mathscr{F}\ell} = G^\vee/U^\vee$ and $\mathscr{F}\ell = G^\vee/B^\vee$ be the enhanced flag variety and flag variety, respectively. The natural (right) $T^\vee$-action on $\widetilde{\mathscr{F}\ell}$ makes the projection $\pi\colon \widetilde{\mathscr{F}\ell} \to \mathscr{F}\ell$ a $G^\vee$-equivariant $T^\vee$-torsor.

For each $w \in W$, put

$$\widetilde{\mathscr{F}\ell}_w = \pi^{-1}(\mathscr{F}\ell_w).$$

The $T^\vee$-torsor $\pi\colon \widetilde{\mathscr{F}\ell}_w \to \mathscr{F}\ell_w$ is trivial. For each $w \in W$, put

$$\widetilde{M}_w = \tilde{i}_{w!}\mathscr{E}[\ell(w)],$$

where $\tilde{i}_w\colon \widetilde{\mathscr{F}\ell}_w \hookrightarrow \widetilde{\mathscr{F}\ell}$ is the inclusion, and $\mathcal{E}$ denotes the free pro-unipotent local system on $\mathscr{F}\ell_w$. That is, $\mathcal{E}$ is the local system corresponding to representation of the group algebra of $\pi_1(T^\vee) = \pi_1(\widetilde{\mathscr{F}\ell}_w)$ obtained by completion along the augmentation ideal. The $\widetilde{M}_w$ are pro-objects in $D(U^\vee\backslash\widetilde{\mathscr{F}\ell})$. The convolution formalism extends to these pro-objects, and we have

$$\text{if } \ell(vw) = \ell(v) + \ell(w), \text{ then } \widetilde{M}_v \cdot \widetilde{M}_w = \widetilde{M}_{vw}.$$

Moreover, the $\widetilde{M}_w$ are invertible, with inverse $\widetilde{M}_w^{-1} = \tilde{i}_{w^{-1}*}\mathcal{E}[\ell(w^{-1}])$.

0.5. **Category $\mathcal{O}$.** Let $\mathcal{O}_0 \subsetneq \mathrm{Perv}(\mathscr{F}\ell)$ be the full subcategory consisting of $U^\vee$-equivariant sheaves. As $U^\vee$ is contractible, this is the same as the subcategory of $U^\vee$-monodromic sheaves. Further, as $U^\vee$-orbits and $B^\vee$-orbits on $\mathscr{F}\ell$ coincide, this is also the subcategory of $B^\vee$-monodromic sheaves. I.e., perverse sheaves smooth along the stratification $\mathscr{F}\ell = \bigsqcup_{w\in W} \mathscr{F}\ell_w$. The natural functor from $D(\mathcal{O}_0)$, the bounded derived category of $\mathcal{O}_0$, to $D(\mathscr{F}\ell)$ is full and faithful. So we can and will identify $D(\mathcal{O}_0)$ with a full subcategory of $D(\mathscr{F}\ell)$.

Convolution formalism yields functors $\widetilde{M}_w \cdot -\colon D(\mathcal{O}_0) \to D(\mathcal{O}_0)$. A priori, convolving with the pro-objects $\widetilde{M}_w$ yields pro-objects in $D(\mathcal{O}_0)$. However, one may check that these are actually honest objects of $D(\mathcal{O}_0)$. Furthermore, if $L \in \mathrm{Perv}(B\backslash\mathscr{F}\ell) \subset D(\mathcal{O}_0)$, then

$$\widetilde{M}_w \cdot L = \mathbf{T}_w \cdot L[\ell(w)].$$

It follows that the $\widetilde{M}_w \cdot -$ (resp. $\widetilde{M}_w^{-1}$) are left (resp. right) t-exact.

0.6. **Free monodromic tilting sheaves.** Let $\mathscr{F}\ell_{\leq w}$ (resp. $\widetilde{\mathscr{F}\ell}_{\leq w}$) denote the closure of $\mathscr{F}\ell_w$ (resp. $\widetilde{\mathscr{F}\ell}_w$) in $\mathscr{F}\ell$ (resp. $\widetilde{\mathscr{F}\ell}$). For each simple reflection $s \in W$ one may find a U-equivariant regular function f on $\widetilde{\mathscr{F}\ell}_{\leq s}$ such that $f^{-1}(0) = \widetilde{\mathscr{F}\ell}_{<s} = \widetilde{\mathscr{F}\ell}_e$. Set

$$\widetilde{\mathcal{T}}_s = \Xi_f(\mathcal{E}[1]),$$

where $\Xi_f \colon \mathrm{Perv}(\widetilde{\mathscr{F}\ell}_s) \to \mathrm{Perv}(\widetilde{\mathscr{F}\ell}_{\leq s}) \hookrightarrow \mathrm{Perv}(\widetilde{\mathscr{F}\ell})$ is Beilinson's maximal extension functor. Then $\widetilde{\mathcal{T}}_s$ is Verdier self-dual, and we have a short exact sequence of pro-objects:

$$0 \to \widetilde{M}_s \to \widetilde{\mathcal{T}}_s \to \psi_f(\mathcal{E}[1]) \to 0,$$

where ψ_f denotes the (unipotent part of) nearby cycles. We infer that $\widetilde{\mathcal{T}}_s$ is an indecomposable tilting sheaf (with free pro-unipotent monodromy).

Let $\mathcal{P}$ be the smallest subcategory of (pro-)perverse sheaves on $\widetilde{\mathscr{F}\ell}$ containing $\widetilde{\mathcal{T}}_s$, for each simple reflection s, and closed under taking direct summands. This is the eponymous category of projective functors.

References

[BG] A. Beilinson, V. Ginzburg, *Wall-crossing functors and $\mathcal{D}$-modules*, Rep. Theory **3** (1999), 1-31 (electronic).