NORMAL CROSSINGS DIVISORS

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1. **Some sheaf theory.** Throughout all sheaves will be abelian sheaves. For once all functors *will not* be derived. That is, if I write f_* I mean just f_*, right derived functors will be preceded by an 'R'. I hope what I mean by this is clear from the formula '$f_* = R^0 f_*$'.

Now suppose $i\colon Z \hookrightarrow X$ is a closed immersion. Then set

$$\Gamma_Z(-) = i_* i^!(-).$$

Here $i^!$ is the left exact functor of 'sections with support in Z'. This functor is right adjoint to i_*, in particular we have an adjunction map $i_* i^! \to \mathrm{id}$ (on stalks these are the obvious 'inclusion' maps). Note: as the left adjoint of $i^!$, namely i_*, is exact, $i^!$ preserves injective objects.

The following is straightforward (i.e., I leave the proof to you).

Proposition 1.1. *Let F be a sheaf on X, and let $Z_1, Z_2 \subseteq X$ be closed subspaces that cover X. Then the sequence*

$$0 \to \Gamma_{Z_1 \cap Z_2}(F) \xrightarrow{+} \Gamma_{Z_1}(F) \oplus \Gamma_{Z_2}(F) \xrightarrow{-} F$$

is exact. Here $+ = (\eta_1, \eta_2)$, $- = (\eta_1', -\eta_2')$, and $\eta_i\colon \Gamma_{Z_i}(F) \to F$, $\eta_i'\colon \Gamma_{Z_1 \cap Z_2}(F) \to \Gamma_{Z_i}(F)$ are the adjunction maps.

Recall that a sheaf F on X is called flabby if the restriction $F(X) \to F(U)$ is surjective for each open subset $U \subset X$. Injective sheaves are flabby. Once more I leave the proof of the following to you:

Proposition 1.2. *Let F be a flabby sheaf on X, and let $Z_1, Z_2 \subseteq X$ be closed subspaces that cover X. Then the sequence*

$$0 \to \Gamma_{Z_1 \cap Z_2}(F) \xrightarrow{+} \Gamma_{Z_1}(F) \oplus \Gamma_{Z_2}(F) \xrightarrow{-} F \to 0$$

is exact. The maps $+$ and $-$ are as before.

It should now come as no surprise that if we have closed subspaces $Z_1, \ldots, Z_n \subseteq X$ which cover X, then (for F flabby) we obtain a 'generalized Mayer-Vietoris' exact sequence:

$$\cdots \to \bigoplus_{i<j<k} \Gamma_{Z_i \cap Z_j \cap Z_k}(F) \to \bigoplus_{i<j} \Gamma_{Z_i \cap Z_j}(F) \to \bigoplus_i \Gamma_{Z_i}(F) \to F \to 0.$$

The proof of this, as with all these sort of simplicial gadgets, is a combinatorial pain in the neck. One day I am going to figure out a clean way of writing it. Needless to say I am not going to write up a proof here. There is a pedantic comment I need to make here though: technically the direct sums above should be direct products, but since we are only dealing with finitely many objects both of these coincide.

2. **Normal crossing divisors.** Let X be a smooth variety of dimension d, and $D = \sum_{i=1}^{n} D_i$ be a strict normal crossings divisor (so in particular I am assuming each D_i is smooth). Set

$$D_i = \bigcup_{\alpha_1 < \cdots < \alpha_i} D_{\alpha_1} \cap \cdots \cap D_{\alpha_i},$$

$$D(i) = \bigsqcup_{\alpha_1 < \cdots < \alpha_i} D_{\alpha_1} \cap \cdots \cap D_{\alpha_i}.$$

So D_i is the locus of points ('of multiplicity $\geq i$') contained in at least i irreducible components, and $D(i)$ is its normalization. Note that $D(i)$ is of dimension $d - i$. Write $v_i \colon D(i) \to D$ for the evident map. Let $i \colon D \hookrightarrow X$ be the inclusion. Set $U = X - D$, and write $j \colon U \hookrightarrow X$ for the corresponding inclusion. Note that if I is an injective sheaf on X, then we have a short exact sequence

$$0 \to i_* i^!(I) \to I \to j_* j^*(I) \to 0.$$

So, for I injective,

$$0 \to i_* i^!(I) \to I$$

is an injective resolution of $j_* j^*(I)$ (note that i_* preserves injectives, since it has an exact left adjoint). Further, applying our generalized Mayer-Vietoris sequence to $i^!(I)$ we obtain an exact sequence

$$\cdots \to v_{2*} v_2^!(i^!(I)) \to v_{1*} v_1^!(i^!(I)) \to i^! I \to 0.$$

In other words,

$$\cdots \to v_{2*} v_2^!(i^!(I)) \to v_{1*} v_1^!(i^!(I))$$

is an injective resolution of $i^!(I)$. Set $u_i = i \circ v_i$. Then putting all of the above together we get that

$$\cdots \to u_{2*} u_2^!(I) \to u_{1*} u_1^!(I) \to I$$

is an injective resolution of $j_* j^* I$.

Now suppose $I^\bullet$ is an injective resolution of the constant sheaf $\mathbf{1}_X$. Then the total complex of the double complex

$$\cdots \to u_{2*} u_2^!(I^\bullet) \to u_{1*} u_1^!(I^\bullet) \to I^\bullet$$

is quasi-isomorphic to $R j_* \mathbf{1}_U$. Now $u_i^!(I^\bullet)[2n]$ is quasi-isomorphic to the dualizing complex on $D(i)$. But $D(i)$ is smooth, so we have that $u_i^!(I^\bullet)$ is quasi-isomorphic to $\mathbf{1}_{D(i)}[-2i]$.

Set $I_i^\bullet = u_i^* I^\bullet$. Then $I_i^\bullet$ is an injective resolution of the constant sheaf on $D(i)$. Thus, the total complex of the double complex

$$\cdots \to u_{2*} I_2^\bullet[-4] \to u_{1*} I_1^\bullet[-2] \to I^\bullet$$

is an injective resolution of $R j_* \mathbf{1}_U$. It follows that

$$R^p j_* \mathbf{1}_U \simeq u_{p*} \mathbf{1}_{D(p)}.$$

If we had been doing this in any sort of 'mixed setting' with Tate twists, then you need to put a '(i)' beside each '$[2i]$'. In such a setting the last formula becomes

$$R^p j_* \mathbf{1}_U \simeq u_{p*} \mathbf{1}_{D(p)}(-p).$$

I'll leave it to you make the appropriate modifications when dealing with other lisse sheaves.

3. **Degeneration of Leray.** Let's assume that we are now working with schemes over a finite field, and all of the above makes sense for ℓ-adic sheaves. That is, we have an isomorphism

$$R^p j_* \underline{\mathbf{Q}_\ell}_U \simeq u_{p*} \underline{\mathbf{Q}_\ell}_{D(p)}(-p)$$

which is compatible with the Frobenius action. Then we have the Leray spectral sequence with

$$E_2^{p,q} = H^p(D(q))(-q) \text{ converging to } H^{p+q}(U).$$

Here cohomology is with $\mathbf{Q}_\ell$-coefficients. Now as $D(q)$ is smooth and projective, by the Weil conjectures (Riemann hypothesis part), $H^p(D(q))(-q)$ is pure of weight $2q+p$. Now the differentials in our spectral sequence are compatible with weights, and the differential on the E_2 page is a map

$$H^p(D(q))(-q) \to H^{p+2}(D(q-1))(-q+1).$$

The right hand side has weight $p+2+2(q-1) = p+2q$. So $E_3^{p,q}$ has weight $2q+p$. Thus, the differential on the E_3-page consists of maps from weight $2q+p$ objects to weight $2(q-2)+p+3 = 2q+p-1$ objects. Hence, it must be 0. Thus, the Leray spectral sequence degenerates at E_3.

The same argument applies for a complex variety. Well, that is if you believe that you can make the differentials in the Leray spectral sequence compatible with the Hodge structures on the terms (and this is pretty much, in my opinion, the whole point of Deligne's Hodge II).

Of course if you squint at the above a bit more, you see that the weight d part of $H^d(U)$ is precisely the image of $H^d(X) \to H^d(U)$. This pretty much gives you the Theorem of the fixed part.

Anyway, my point is that the argument is basically the same in the ℓ-adic setting and the complex varieties setting (in fact, by just using the words 'pure' and 'weights' the proof can be made identical). Of course, before Hodge II it wasn't clear that you could put a Hodge structure on $H^*(U)$. On the other hand, all the statements were known in the ℓ-adic setting (granting the Weil conjectures). Now it was also known that $H^*(U)$ would have mixed Frobenius weights. So if you believe in motivic philosophy, the main hurdle that needed to be crossed was to come up with a notion of mixed for Hodge structures. Even here actually you can look at the Frobenius setting and come up with the definition.

Now for all these weight arguments you either need $\mathbf{Q}_\ell$ coefficients, or in the case of complex varieties $\mathbf{C}$ coefficients. So if you want a statement for $\mathbf{Z}_\ell$ coefficients, you are going to either need a lot more geometric information: you need to know that all the E_3-terms in your spectral sequence are torsion free. This would require a lot of non-trivial work I would think, since it's not enough for your E_2-terms to be torsion free. It is a virtual guarantee that the degeneration doesn't hold in general for $\mathbf{Z}_\ell$-coefficients. However, I'd have to think a bit about a specific example.

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