For the relative trace for motivic sheaves we need some 'bounds on motivic cohomology'. Instead of trying to be cute with weight structures, let me just do everything from scratch. Some parts are overly pedantic - I am trying to make sure I don't mess up in any significant way.

As usual, I will work with complex varieties, $\underline{X}$ is the constant sheaf, etc. The category we work with now though is motivic sheaves.

Lemma. *If X is smooth, then*

$$\mathrm{Hom}(\underline{X}, \underline{X}[n](m)) = 0 \quad \textit{for } m < 0.$$

Proof. We have

$$\mathrm{Hom}(\underline{X}, \underline{X}[n](m)) = CH^m(X, 2m - n).$$

(This is p. xxi of Cisinski-Deglise, arXiv:0912.2110v4). Now we recall the definition of the higher Chow groups. For a natural number j, the algebraic j-simplex Δ^j is defined to be the hyperplane $x_0 + \cdots + x_j = 1$ in $\mathbf{C}^{j+1}$. For an equidimensional variety X, one defines $z^i(X, j)$ as a certain subgroup of the free abelian group generated by codimension-i subvarieties of $X \times \Delta^j$. Then there is a chain complex

$$\cdots \to z^i(X, 2) \to z^i(X, 1) \to z^i(X, 0) \to 0,$$

and the higher Chow group $CH^i(X, j)$ is the homology of this chain complex at $z^i(X, j)$. In particular, $CH^i(X, j) = 0$ for $i < 0$, since there are no subvarieties of negative codimension. □

Remark. The t-structure conjectures are that these groups vanish for $n < 0$.

Given a morphism $f\colon X \to S$, it will be notationally convenient to set:

$$H_{a,b}(X/S) = \mathrm{Hom}(f_!\underline{X}, \underline{S}[-a](-b)),$$
$$d_{X/S} = \text{maximum of the dimension of the fibers of } f.$$

Now suppose $j\colon U \to X$ is an open immersion and let $i\colon Z \to X$ be the inclusion of the closed complement. Then we have the localization triangle

$$f_!j_!\underline{U} \to f_!\underline{X} \to f_!i_*\underline{Z} \xrightarrow{+1}$$

Applying $\mathrm{Hom}(-, \underline{S}[-a](-b))$ to this yields an exact sequence

$$\cdots \to H_{a,b}(Z/S) \to H_{a,b}(X/S) \to H_{a,b}(U/S) \to H_{a-1,b}(Z/S) \to \cdots$$

Similarly, working with the base now, if $j\colon U \to S$ is an open immersion with $i\colon Z \to S$ the closed complement, then we may form a commutative diagram with cartesian squares:

$$\begin{array}{ccccc} X_U & \longrightarrow & X & \longrightarrow & X_Z \\ \downarrow & & \downarrow f & & \downarrow \\ U & \xrightarrow{j} & S & \xrightarrow{i} & Z \end{array}$$

Then applying $\mathrm{Hom}(f_!\underline{X}, -)$ to the localization triangle

$$j_!j^*\underline{S} \to \underline{S} \to i_*i^*\underline{S} \xrightarrow{+1}$$

and using proper base change, yields an exact sequence

$$\cdots \to H_{a,b}(X_U/U) \to H_{a,b}(X/S) \to H_{a,b}(X_Z/Z) \to H_{a-1,b}(X_U/U) \to \cdots$$

Neither the index a (the 'cohomological index') nor the direction of the maps in the above exact sequences is important for us. What matters is that we can 'squeeze' X (or S) between U and Z, and that the 'weight index' b stays constant. If we were in the (mixed) Hodge setting these would be exact sequences associated to a fixed weight space. Roughly, the point is that even though in the motivic setting we don't have vanishing bounds on the cohomological index (t-structure conjectures) we do have bounds on weights.

Proposition. *One has*

$$H_{a,b}(X/S) = 0 \quad \text{whenever } b > d_{X/S}.$$

Proof.
Step 1: assume both S and $f\colon X \to S$ are smooth (in particular X is smooth).

In this case,

$$\begin{aligned} H_{a,b}(X/S) &= \mathrm{Hom}(f_!\underline{X}, \underline{S}[-a](-b)) \\ &= \mathrm{Hom}(\underline{X}, f^!\underline{S}[-a](-b)) \\ &= \mathrm{Hom}(\underline{X}, f^*\underline{S}[-a+2d_{X/S}](-b+d_{X/S})) \\ &= \mathrm{Hom}(\underline{X}, \underline{X}[-a+2d_{X/S}](-b+d_{X/S})) \end{aligned}$$

So the Lemma gives the desired result.

Step 2: assume S is smooth, but $f\colon X \to S$ is arbitrary.

Proceed by induction on $\dim(X)$. Step 1 yields the base of our induction. So assume the statement is true for all varieties with dimension less than $\dim(X)$. We may assume X is irreducible. Let $U \subset X$ be the (open dense) locus of points on which f is smooth. Write $Z = X - U$ for the closed complement. Then we have an exact sequence

$$H_{a,b}(Z/S) \to H_{a,b}(X/S) \to H_{a,b}(U/S).$$

By Step 1, $H_{a,b}(U/S)$ vanishes for $b > d_{X/S}$. By the induction hypothesis, $H_{a,b}(Z/S)$ vanishes for $b > d_{X/S}$.

Step 3: now suppose S is arbitrary.

Proceed by induction on $\dim(S)$. Step 2 yields the base of our induction. So assume the statement is true for all varieties with dimension less than $\dim(S)$. Let $U \subset S$ be the smooth locus, and write $Z = S - U$ for the closed complement. Then we have an exact sequence

$$H_{a,b}(X_U/U) \to H_{a,b}(X/S) \to H_{a,b}(X_Z/Z).$$

By Step 2, $H_{a,b}(X_U/U)$ vanishes for $b > d_{X/S}$. By the induction hypothesis, $H_{a,b}(X_Z/Z)$ vanishes for $b > d_{X/S}$. $\square$

Now we seek a canonical map (relative trace/integration along the fiber):

$$f_! \underline{X} \to \underline{S}[-2d_{X/S}](-d_{X/S}).$$

Assume X is irreducible. Let $j\colon U \to X$ be the open dense locus of points over which f is smooth. Let $i\colon Z \to X$ be the closed complement. As

$$\begin{aligned}\mathrm{Hom}(f_! j_! \underline{U}, \underline{S}[-2d_{X/S}](-d_{X/S})) &= \mathrm{Hom}(\underline{U}, (f \circ j)^! \underline{S}[-2d_{X/S}](-d_{X/S})) \\ &= \mathrm{Hom}(\underline{U}, (f \circ j)^* \underline{S}) \\ &= \mathrm{Hom}(\underline{U}, \underline{U}),\end{aligned}$$

it suffices to show that the canonical map

$$\mathrm{Hom}(f_! \underline{X}, \underline{S}[-2d_{X/S}](-d_{X/S})) \to \mathrm{Hom}(f_! j_! \underline{U}, \underline{S}[-2d_{X/S}](-d_{X/S}))$$

is an isomorphism. In our notation, we need to show that the canonical map

$$H_{2d_{X/S}, d_{X/S}}(X/S) \to H_{2d_{X/S}, d_{X/S}}(U/S)$$

is an isomorphism. But now we have our exact sequence:

$$H_{2d_{X/S}, d_{X/S}}(Z/S) \to H_{2d_{X/S}, d_{X/S}}(X/S) \to H_{2d_{X/S}, d_{X/S}}(U/S) \to H_{2d_{X/S}-1, d_{X/S}}(Z/S).$$

The outer groups vanish by the Proposition, since the dimension of the fibers of $Z \to S$ are strictly less than $d_{X/S}$. So we are done.