Dear Matthias and Wolfgang,

Here is a construction of the six functors. No messiness with 'pseudo-adjoints'.

1. Initial assumptions:
   (i) For varieties, categories $DM(X)$ have been defined. These are triangulated, compactly generated, admit arbitrary sums and the six functor formalism.
   (ii) For an algebraic group G, the categories $DM_G(X)$ have been defined. These are triangulated, admit arbitrary sums and the functors $\otimes$, f^*, $g_\#$ ($g_\#$ is a left adjoint to g^* for smooth g). These satisfy those aspects of the six functor formalism that their analogues do in the non-equivariant setting. Further, $*$-pullback along surjective maps is conservative. General $*$-pullback is required to commute with arbitrary direct sums.
   (iii) If G is smooth, then a quotient equivalence has been defined which is compatible with the above available functors.

I will comment on (ii) and (iii) and our simplicial setting at the end.

2. **Forgetful functor.** Let $\alpha\colon G \times X \to X$ be a G-action with G smooth. Define

$$\mathrm{For}\colon DM_G(X) \to DM(X)$$

to be the composition $DM_G(X) \xrightarrow{\alpha^*} DM_G(G \times X) \xrightarrow{\sim} DM(X)$, where G acts on $G \times X$ by multiplication on the first factor and trivially on the second. Then For is conservative and commutes with direct sums.

3. **#-integration.** Keep the above notation. Then we have a left adjoint to For,

$$\Gamma^G_\#\colon DM(X) \to DM_G(X),$$

given by the composition $DM(X) \xrightarrow{\sim} DM_G(G \times X) \xrightarrow{\alpha_\#} DM_G(X)$.

4. **Compact generation.** The following will get things going.

Lemma 4.1. *For smooth G, the category $DM_G(X)$ is compactly generated.*

Proof. Let $\mathcal{P}$ be a set of compact generators for $DM(X)$. Let

$$\Gamma\mathcal{P} = \{\Gamma^G_\#(P) \mid P \in \mathcal{P}\}.$$

Now $\mathrm{Hom}_{DM_G(X)}(\Gamma^G_\#(P), M) \simeq \mathrm{Hom}_{DM_G(X)}(P, \mathrm{For}(M))$. Thus, the left hand side being zero for all $P \in \mathcal{P}$ implies $\mathrm{For}(M) = 0$. As For is conservative, $M = 0$. So $\Gamma\mathcal{P}$ is a set of generators for $DM_G(X)$. For a sum $\bigoplus_i M$ in $DM_G(X)$,

$$\begin{aligned}
\mathrm{Hom}_{DM_G(X)}(\Gamma^G_\#(P), \bigoplus_i M) &\simeq \mathrm{Hom}_{DM(X)}(P, \mathrm{For}(\bigoplus_i M)) \\
&\simeq \mathrm{Hom}_{DM(X)}(P, \bigoplus_i \mathrm{For}(M_i)) \\
&\simeq \bigoplus_i \mathrm{Hom}_{DM(X)}(P, \mathrm{For}(M_i)) \\
&\simeq \bigoplus_i \mathrm{Hom}_{DM_G(X)}(\Gamma^G_\#(P), M_i).
\end{aligned}$$

Here the second isomorphism is the commutativity of For with sums, and the third isomorphism utilizes the compactness of P. So the $\Gamma^G_\#(P)$ are compact. □

5. $*$**-pushforward.** As $DM_G(X)$ is compactly generated and f^* is assumed to commute with direct sums, Neeman's Brown Representability Theorem yields a right adjoint f_* to f^*. This behaves as expected with $\otimes$ (by adjunction). If g is a smooth map, then f_* and g^* satisfy smooth base change (whenever this makes sense). This is so because the map in the statement of smooth base change is obtained by the adjunctions between $*$-pullback/pushforward. It is an isomorphism because it is so after applying the conservative functor For.

6. !**-pushforward.** Call an equivariant map $f\colon X \to Y$ compactifiable if it factors as $f = p \circ j$, where j is an equivariant open immersion and p is proper and equivariant. Certainly, if X is quasi-projective and our action is linearizable, then each f is compactifiable. For compactifiable f, set $f_! = p_* \circ j_\#$. This is independent of the compactification by the usual argument (see the construction of $f_!$ in the étale setting). It satisfies all the expected properties, since p_* and $j_\#$ do. This might feel a bit unsettling, especially the compactifiable part. However, I don't think we should worry about this since, as far as I understand it, !-pushforward is available from the beginning in our setting (although, note that in the motivic literature, or the étale setup, this is exactly how !-pushforward is constructed in the non-equivariant setting). I am basically cheating and saying that if this is a worry, just throw its existence into the starting assumptions.

7. !**-pullback.** To apply Neeman's result, we need:

Lemma 7.1. *The functor $f_!$ commutes with direct sums.*

Proof. Given a direct sum $\bigoplus_{i\in I} M_i$, for each $j \in I$, we have a canonical map $M_j \to \bigoplus_i M_i$. So for each j we get a canonical map $f_! M_j \to f_! \bigoplus_{i\in I} M_i$. This induces a canonical map $\bigoplus_{i\in I} f_! M_i \to f_! \bigoplus_{i\in I} M_i$. This map is an isomorphism after applying For (by construction, our $f_!$ is compatible with For). As For is conservative, it was an isomorphism to begin with. □

So we get a right adjoint to $f_!$, with the expected properties (by adjunction).

8. **Duality.** I haven't checked the details for this, but I believe one should just define dualizing objects via $f^!$ and check this indeed gives a duality (on compact objects) by applying For and using conservativity.

9. **Localization triangles.** I need to show that the canonical adjunction maps, for i a closed immersion and j the complementary open immersion, fit into distinguished triangles

$$i_* i^! \to \mathrm{id} \to j_* j^* \quad \text{and} \quad j_! j^* \to \mathrm{id} \to i_* i^*$$

I am going to only deal with the first one, the argument for the second one is similar. For an arbitrary M, let C_M denote the cone of $i_* i^! M \to M$. Then using For and conservativity one sees that $j^* C_M \xrightarrow{\sim} j^* M$. Now take the cone of $C_M \to j_* j^* C_M$ and apply For again to get that $C_M \xrightarrow{\sim} j_* j^* M$. Using the usual properties of adjunction maps, one gets that the composition $M \to C_M \xrightarrow{\sim} j_* j^* M$ is precisely the adjunction map.

10. **Some simplicial spaces.** We need to get properties (i), (ii) and (iii) in a simplicial setting. Mainly property (iii). Out of necessity the standard simplicial players make an appearance.

First, any variety S can be treated as a 'constant' simplicial variety. Next, for a G-variety X, we have the simplicial space $[X/G]$. Finally, given a map of varieties $X \to S$, we have the simplicial variety given by taking iterated fibre products along the base change $X \to S$ (Čech cover style). Denote this simplicial variety by $[X/S]$.

As usual, a G-torsor means an equivariant faithfully flat map $X \to S$ (G acting on S trivially) such that the canonical map $G \times X \to X \times_S X$ is an isomorphism. Most relevant for us is that for a G-torsor $X \to S$, the simplicial variety $[X/S]$ is canonically isomorphic to $[X/G]$, pretty much by definition. Further, if G is smooth, then $X \to S$ admits étale local sections.

11. **Simplicial categories.** Now we start from scratch. That is, discard the old notation and assumptions about $DM_G(X)$, $DM(X)$. Instead take as starting point the following assumptions (using which I will construct the categories satisfying the original assumptions):

- (i) To each simplicial variety $X_\bullet$ we have an assignment of a triangulated category $DM(X_\bullet)$ that admits arbitrary direct sums and the formalism of the functors $\otimes$, f^*, $g_\#$.
- (ii) f^* commutes with arbitrary sums. It is conservative for surjective f.
- (iii) **(Cartesian)** Each $DM(X_\bullet)$ contains a localizing subcategory

$$DM^\Delta(X_\bullet) \subset DM(X_\bullet)$$

  with the following properties:
  - (a) The $DM^\Delta(X_\bullet)$ are stable under $\otimes$, f^*, $g_\#$.
  - (b) For varieties X (viewed as constant simplicial varieties), the categories $DM^\Delta(X)$ admit the full six functor formalism and the functors $\otimes$, f^*, $g_\#$ are compatible with those in (i).
  - (c) For a variety X, the category $DM^\Delta(X)$ is compactly generated.
  - (d) If $X \to S$ is a morphism of varieties and $a\colon [X/S] \to S$ the induced morphism of simplicial varieties, then $DM^\Delta([X/S])$ is the essential image of $a^*\colon DM^\Delta(S) \to DM([X/S])$.
- (iv) **(Descent)** If $X \to S$ is a morphism of varieties admitting étale local sections, and $a\colon [X/S] \to S$ is the induced morphism of simplicial varieties, then

$$a^*\colon DM^\Delta(S) \xrightarrow{\sim} DM^\Delta([X/S])$$

  is an equivalence of categories (equivalently, $a^*\colon DM^\Delta(S) \to DM([X/S])$ is full and faithful).

For a G-variety X, define

$$DM_G(X) = DM^\Delta([X/G]).$$

Note that the ordinary (i.e., non-equivariant) category $DM_1(X)$ is $DM^\Delta(X)$. This notational nuisance occurs because we are treating X as a constant simplicial variety. Once the formalism of the equivariant category is up and running, we can forget about simplicial things and revert to saner notation.

To summarize, we have the full six functor formalism for ordinary varieties. For general simplicial varieties we assume a partial formalism of $\otimes$, f^* and $g_\#$.

I hope it is clear that this is not terribly different from the initial assumptions I first started with. I have just rephrased things for the simplicial setting. We need a quotient equivalence, but this is basically just **(Descent)**:

Proposition 11.1 (Quotient equivalence)**.** *Let X be a G-variety for a smooth algebraic group G. If $X \to S$ is a G-torsor, then*

$$DM_G(X) = DM^\Delta([X/G]) \simeq DM^\Delta(S).$$

This equivalence is compatible with $\otimes$, f^* and $g_\#$.

Proof. By definition of a G-torsor, $[X/G] = [X/S]$. As G is smooth, $X \to S$ is smooth and hence admits étale local sections. So **(Descent)** yields the equivalence. Compatibility with $\otimes$ and f^* is obvious. For $g_\#$ it follows by adjunction. □

(Descent) will probably be the hardest thing to prove in our situation. I don't know how to do it. This involves a deeper look into the motivic machinery and doesn't follow from simplistic formal arguments as above. We could just aim to prove the special case of the quotient equivalence directly. I doubt that this will be any easier. Even in the setting of ordinary sheaves, these statements (in the simplicial setting) are not quite trivial.

The **(Cartesian)** property is also a bit subtle in our situation. Namely, if we just take naïve 'Cartesian' objects (all structure maps isomorphisms for 'simplicial motives') in $DM([X/G])$, these are not the correct things to consider - this category is too big (it is essentially the naïvely equivariant objects of the ordinary derived category). An essentially equivalent perspective is that this is the standard problem with descent data for derived categories. However, I think it should be possible to cheat and just use property (d) in **(Cartesian)** as the definition.

Best,
rv

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The Appalachians