SOME NOTIONS FROM COMMUTATIVE ALGEBRA

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All rings will be commutative with 1.

1. Algebras

1.1. Let $f\colon A \to B$ be a ring homomorphism. If $a \in A$ and $b \in B$, define a product

$$ab = f(a)b.$$

This definition of scalar multiplication makes the ring B into an A-module. Thus, B has an A-module structure as well as a ring structure. These structures are compatible in an obvious sense. The ring B, equipped with this A-module structure, is said to be an A-*algebra*. So, an A-algebra is, by definition, a ring B together with a ring homomorphism $f\colon A \to B$.

1.2. *Example.* Let k be a field. Then a k-algebra is effectively a ring containing k as a subring. For instance, the polynomial ring $k[x_1, \ldots, x_n]$.

1.3. Let B, B' be A-algebras. A morphism of A-algebras or an A-algebra homomorphism $\phi\colon B \to B'$ is a ring homomorphism which is also an A-module homomorphism.

1.4. A ring homomorphism $f\colon A \to B$ is *finite*, and B is said to be finite over A, if B is finitely generated as an A-module.

1.5. A ring homomorphism $f\colon A \to B$ is of *finite type*, and B is said to be a finitely generated algebra over A, if there exists a finite set of elements $x_1, \ldots, x_n$ in B such that every element of B can be written as a polynomial in $x_1, \ldots, x_n$ with coefficients in $f(A)$. This is equivalent to requiring a surjective A-algebra morphism from a polynomial ring $A[t_1, \ldots, t_n]$ onto B.

2. Finite vs. integral

2.1. The following result and its proof should be reminiscient of Nakayama's lemma (actually Nakayama's lemma is a special case of this result).

2.2. **Lemma** (Determinant trick)**.** *Let A be a ring and M an $A[t]$-module that is finitely generated as an A-module. Suppose $\mathfrak{a}$ is an ideal of A such that $A[t]M \subseteq \mathfrak{a}M$. Then the action of t on M satisfies a relation of the form*

$$t^n + a_1 t^{n-1} + \cdots + a_n = 0,$$

where each $a_i \in \mathfrak{a}^i$.

Proof. Let $v_1, \ldots, v_n$ be a set of generators for M. As $A[t]M \subseteq \mathfrak{a}M$ we obtain equations of the form

$$tv_i = \sum_j a_{ij} v_j, \qquad \text{with } a_{ij} \in \mathfrak{a}.$$

These can be rewritten as

$$\sum_j (a_{ij} - \delta_{ij} t) v_j = 0.$$

Let T be the $n \times n$-matrix with (i, j)-th entry $(a_{ij} - \delta_{ij} t)$. Then the determinant of this matrix gives an expression of the required form. □

2.3. Let B be an A-algebra. An element $y \in B$ is *integral* over A if there exists a *monic* polynomial $f(x) \in A[x]$ such that $f(y) = 0$. The algebra B is *integral* over A (or B is an *integral extension* of A) if every $b \in B$ is integral.

2.4. *Example.* Let $F \supset K$ be a field extension. Then F is integral over K if and only if F is algebraic over K.

2.5. **Proposition.** *Let B be an A-algebra and let $y \in B$. The following conditions are equivalent:*

(i) *y is integral over A.*
(ii) *The subring $A[y] \subseteq B$ generated by A and y is finite over A.*
(iii) *There exists an A-subalgebra $C \subseteq B$ such that $A[y] \subseteq C$ and C is finite over A.*

Proof. That (i) implies (ii) is left as an exercise (Hint: there is a similar statement for field extensions that we proved earlier). That (ii) implies (iii) is obvious. Let's show that (iii) implies (i). The algebra C is an $A[t]$-module via $p(t) \cdot x = p(y)x$, $p(t) \in A[t], x \in C$. As C is finite over A, by the determinant trick we obtain a relation

$$y^n + a_{n-1}y^{n-1} + \cdots + a_0 = 0, \qquad \text{with } a_i \in A.$$ □

2.6. *Remark.* The point of the above result is that for an A-algebra B,

$$\text{finite type} + \text{integral over } A = \text{finite over } A.$$

3. Tower laws

3.1. **Proposition.** *Let B be an A-algebra and let C be a B-algebra (note that this gives an A-algebra structure on C). If C is finite over B and B is finite over C, then C is finite over A.*

Proof. Exercise! Hint: there is a similar statement for field extensions. □

3.2. **Proposition.** *Let B be an A-algebra and let C be a B-algebra (note that this gives an A-algebra structure on C). If C is integral over B and B is integral over A, then C is integral over A.*

Proof. Let $x \in C$. As C is integral over B, we have a relation

$$x^n + b_{n-1}x^{n-1} + \cdots + b_0, \qquad \text{with } b_i \in B.$$

As each b_i is integral over A, $A[b_0, \ldots, b_{n-1}]$ is finite over A by the previous Proposition. By Prop. 2.5, $A[b_0, \ldots, b_{n-1}, x]$ is finite over $A[b_0, \ldots, b_{n-1}]$. Hence, by the previous Proposition, $A[b_0, \ldots, b_{n-1}, x]$ is finite over A. So, by Prop. 2.5, x is integral over A. □

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