IRREDUCIBLE AFFINE VARIETIES, COMPONENTS AND FINITE MORPHISMS

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Contents

Throughout we will work over an algebraically closed ground field k.

1. Closed subsets

1.1. Let X be an affine variety. If Z is a subset of X, set

$$I(Z) = \{p \in k[X] \mid p(z) = 0 \text{ for all } z \in Z\}.$$

If Σ is a subset of $k[X]$, set

$$\text{zeroes}(\Sigma) = \{x \in X \mid p(x) = 0 \text{ for all } p \in \Sigma\}.$$

1.2. A subset $Z \subset X$ is called *closed* if $Z = \text{zeroes}(\Sigma)$ for some subset $\Sigma \subseteq k[X]$.

1.3. *Example.* If X is an algebraic subset of k^n, then closed subsets of X are precisely algebraic subsets $Z \subseteq k^n$ that are contained in X.

1.4. Let $Z \subseteq X$ be a closed subset. Then we can and will define the structure of an affine variety on Z by setting $k[Z] = i^*(k[X])$, where $i\colon Z \hookrightarrow X$ is the inclusion map. It is straightforward (= exercise) to verify that $k[Z] \simeq k[X]/I(Z)$ and that this does indeed endow Z with the structure of an affine variety. It follows trivially that the inclusion $i\colon Z \hookrightarrow X$ is a morphism of affine varieties. From here on any mention of a closed subset as an affine variety is to be understood as just outlined.

1.5. *Remark.* All the results that we proved for the zeroes $-$ I-correspondence in the context of algebraic sets hold in our current setting (the proofs are exactly the same). In our new language we may reformulate the results as follows. Let X be an affine variety. Then:

(i) $\text{zeroes}(0) = X$ and $\text{zeroes}(1) = \emptyset$. In particular, both X and the empty set are closed subsets of X.

(ii) For any family of ideals $\mathfrak{a}_i \subseteq k[X]$, $i \in I$:

$$\text{zeroes}(\bigcup_{i\in I} \mathfrak{a}_i) = \bigcap_{i\in I} \text{zeroes}(\mathfrak{a}_i).$$

In particular, the intersection of any family of closed subsets is closed.

(iii) $\text{zeroes}(\mathfrak{a}\cap\mathfrak{b}) = \text{zeroes}(\mathfrak{a}\mathfrak{b}) = \text{zeroes}(\mathfrak{a})\cup\text{zeroes}(\mathfrak{b})$ for any ideals $\mathfrak{a}, \mathfrak{b} \subseteq k[X]$. In particular, the union of any *finite* family of closed sets is closed.

(iv) Let $\mathfrak{a}, \mathfrak{b} \subseteq k[X]$ be ideals. If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\text{zeroes}(\mathfrak{a}) \supseteq \text{zeroes}(\mathfrak{b})$.
(v) Let $V, Z \subseteq X$ be closed subsets. If $V \subseteq Z$, then $I(V) \supseteq I(Z)$.
(vi) *(Nullstellensatz)* If $\mathfrak{a} \subseteq k[X]$ is an ideal, then $I(\text{zeroes}(\mathfrak{a})) = \sqrt{\mathfrak{a}}$, where $\sqrt{\mathfrak{a}}$ denotes the radical of $\mathfrak{a}$.

1.6. **Proposition.** *Let $f\colon X \to Y$ be a morphism of affine varieties and let $Z \subseteq Y$ be closed. Then*

$$f^{-1}(Z) = \text{zeroes}(f^*(I(Z))).$$

In particular, $f^{-1}(Z)$ is closed in X.

Proof. Exercise! □

1.7. Let X be an affine variety and let Z be a subset of X. The *closure* of Z in X, denoted $\overline{Z}$, is the smallest closed subset of X containing Z. More precisely:

$$\overline{Z} = \bigcap_{\substack{Z \subseteq V, \\ V \text{ closed in } X}} V.$$

It is easy to see that $\overline{Z} = \text{zeroes}(I(Z))$. The subset Z is said to be *dense* in X if $\overline{Z} = X$.

1.8. **Proposition.** *Let $f\colon X \to Y$ be a morphism of affine varieties. Then* $\overline{f(X)} = \text{zeroes}(\ker(f^*))$. *In particular, $f(X)$ is dense in Y if and only if f^* is injective*

Proof. It suffices to show that $I(f(X)) = \ker(f^*)$. Now $p \in I(f(X))$ if and only if $f^*p(x) = p(f(x)) = 0$ for all $x \in X$. I.e., $p \in I(f(X))$ if and only if $f^*p = 0$ □

2. Digression on points and maximal ideals

2.1. Let X be an affine variety. For every $x \in X$, define a k-algebra homomorphism

$$\delta_x\colon k[X] \to k, \quad f \mapsto f(x).$$

Set $\mathfrak{m}_x = \ker(\delta_x)$. Clearly, $\mathfrak{m}_x$ is a maximal ideal. By the Nullstellensatz, the assignment $x \mapsto \mathfrak{m}_x$ gives a bijection

$$\{\text{points of } X\} \xleftrightarrow{1-1} \{\text{maximal ideals in } k[X]\}.$$

2.2. *Example.* Identify $\mathbf{A}^n$ with k^n so that $k[\mathbf{A}^n] = k[x_1, \ldots, x_n]$. Then we get that the point $(a_1, \ldots, a_n) \in \mathbf{A}^n$ corresponds to the maximal ideal $(x_1 - a_1, \ldots, x_n - a_n)$.

2.3. **Proposition.** *Let $f\colon X \to Y$ be a morphism of affine varieties, and let $x \in X$. Then $\mathfrak{m}_{f(x)} = f^{*-1}(\mathfrak{m}_x)$.*

Proof. By definition, $\mathfrak{m}_{f(x)}$ is the kernel of the composition $k[Y] \xrightarrow{f^*} k[X] \xrightarrow{\delta_x} k$. Hence, $\mathfrak{m}_{f(x)} = f^{*-1}(\mathfrak{m}_x)$. □

3. Irreducible varieties

3.1. An affine variety X is called *irreducible* if it is not the union of two proper closed subsets. I.e., if $X = V \cup W$ with $V, W \subseteq X$ closed, then either $V = X$ or $W = X$.

3.2. *Example.* The algebraic set given by the solutions to $xy = 0$ in k^2 is not irreducible (it is the union of the two axis). On the other hand $\mathbf{A}^1$ is certainly irreducible.

3.3. **Proposition.** *X is irreducible if and only if $k[X]$ is an integral domain.*

Proof. Let X be irreducible and let $f, g \in k[X]$ be such that $fg = 0$. Then $X = \text{zeroes}(0) = \text{zeroes}(fg) = \text{zeroes}(f) \cup \text{zeroes}(g)$. As X is irreducible, without loss of generality we may assume that $\text{zeroes}(f) = X$. Then $0 = I(X) = I(\text{zeroes}(f)) = \sqrt{f}$, where $\sqrt{f}$ denotes the radical of the ideal generated by f. As $f \in \sqrt{f}$, we have $f = 0$.

Now let X be an affine variety such that $k[X]$ is an integral domain. Suppose $X = V \cup W$ with $V, W \subseteq X$ closed. Then $0 = I(X) = I(V \cup W) = I(V) \cap I(W)$. As $I(V) \cdot I(W) \subseteq I(V) \cap I(W)$, we infer that $I(V) \cdot I(W) = 0$. But, X is an integral domain. This means that 0 is a prime ideal. Hence, without loss of generality, we may assume that $I(V) = 0$. Therefore, $V = \text{zeroes}(I(V)) = \text{zeroes}(0) = X$. □

4. Components

4.1. **Proposition.** *Let X be an affine variety. Then there exist finitely many irreducible closed subsets $X_1, \ldots, X_n \subseteq X$ such that $X_i \not\subseteq X_j$ for all $i \neq j$ and*

$$X = X_1 \cup \cdots \cup X_n.$$

Moreover, the X_i are unique (up to renumbering of the indices).

Proof. If X is irreducible, then there is nothing to show. Otherwise, $X = V \cup W$ with $V, W \subsetneq X$ proper closed subsets of X. If V and W are finite unions of irreducible closed subsets, then so is X. Thus, if X were not a finite union of irreducible closed subsets, then we could find a closed subset X_1 (either V or W) with the same property. Continuing this way, we would obtain an infinite strictly decreasing chain $X \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots$ of closed subsets X_i. This would yield an infinite strictly ascending chain $0 \subsetneq I(X_1) \subsetneq I(X_2) \subsetneq \cdots$ of ideals in $k[X]$. This is impossible, since $k[X]$ is Noetherian. Hence, $X = X_1 \cup \cdots \cup X_n$, for some (finitely many) irreducible closed subsets $X_i \subseteq X$. Certainly, we may assume that $X_i \not\subseteq X_j$ for all $i \neq j$ (if $X_i \subseteq X_j$, $i \neq j$, just remove X_i from the expression). Now let $X = X'_1 \cup \cdots \cup X'_m$ be another decomposition of this form. We need to show that each X'_i is equal to some $X_{i'}$. As each X'_i is irreducible, it follows (= exercise) that each $X'_i \subseteq X_{i'}$ for some i'. Similarly, each $X_j \subseteq X'_{\overline{j}}$ for some $\overline{j}$. So $X'_i \subseteq X_{i'} \subseteq X'_{\overline{i'}}$, which implies that $i = \overline{i'}$. Consequently, $X'_i = X_{i'}$. □

The X_i appearing in the Proposition above are called the *components* (or *irreducible components*) of X.

4.2. *Example.* The algebraic set given by the solutions to the polynomial $xy = 0$ in k^2 has two components (the x and y axis) each of which is isomorphic to $\mathbf{A}^1$.

5. Finite morphisms

5.1. A morphism of affine varieties $f\colon X \to Y$ is called *finite* if $f^*\colon k[Y] \to k[X]$ is finite. In this situation we say that X is finite over Y. Finite morphisms are quite interesting geometrically. We start with a preliminary result from commutative algebra.

5.2. **Lemma.** *Let B be a commutative ring and let $A \subseteq B$ be a subring. Let $\mathfrak{m}$ be a maximal ideal of A. If B is finite over A, then $\mathfrak{m} = A \cap \mathfrak{m}'$ for some maximal ideal $\mathfrak{m}'$ of B.*

Proof. Exercise! Hint: use Nakayama's lemma to show that $\mathfrak{m}B \neq B$. □

5.3. **Proposition.** *Let $f\colon X \to Y$ be a finite morphism of affine varieties.*

(i) *If $Z \subseteq X$ is closed, then $f(Z)$ is closed.*
(ii) *f is surjective if and only if f^* is injective.*
(iii) *For all $y \in Y$, $f^{-1}(y)$ is a finite set.*

Proof. (i) Let $i\colon Z \hookrightarrow X$ be the inclusion map. If $f^*\colon k[Y] \to k[X]$ is finite, then so is the composition $k[Y] \xrightarrow{f^*} k[X] \xrightarrow{i^*} k[Z]$. Hence, we may assume that $Z = X$. We will now show that $f(X) = \text{zeroes}(\ker(f^*))$. Via the identification of points with maximal ideals (see §2), $f(X)$ consists of the maximal ideals $f^{*-1}(\mathfrak{m})$ as $\mathfrak{m}$ runs through the maximal ideals of $k[X]$, while zeroes($\ker(f^*)$) consists of maximal ideals of $k[Y]$ containing $\ker(f^*)$. Hence, it is clear that $f(X) \subseteq \text{zeroes}(\ker(f^*))$. To show that $\text{zeroes}(\ker(f^*)) \subseteq f(X)$, we need to demonstrate that any maximal ideal of $k[Y]$ containing $\ker(f^*)$ is of the form $f^{*-1}(\mathfrak{m})$ for some maximal ideal $\mathfrak{m} \subseteq k[X]$. This follows from Lemma 5.2.

(ii) Using (i), $f(X) = \overline{f(X)} = \text{zeroes}(\ker(f^*))$. Whence the result.

(iii) If $f^{-1}(y) = \emptyset$, then there is nothing to show. Otherwise, as for (i), we may assume that $X = f^{-1}(y)$. Let $i\colon \{y\} \hookrightarrow Y$ be the inclusion map. Then f is the composition $X \xrightarrow{a} \{y\} \xrightarrow{i} Y$, where a is the obvious map. As f^* is finite, we infer that $a^*\colon k \to k[X]$ is finite. That is, $k[X]$ is a finite dimensional k-vector space. A k-algebra that is finite dimensional as a k-vector space has only finitely many maximal ideals (exercise!). □

5.4. **Warning.** If $f\colon X \to Y$ is a morphism of affine varieties such that $f^{-1}(y)$ is a finite set, then it is *not* generally true that f is finite. For instance, consider the projection of the hyperbola $xy = 1$ (in k^2) on to the x-axis.

6. Geometric form of Noether normalization

6.1. Let X be an affine variety. By Noether Normalization, there exists k-subalgebra $A \subseteq k[X]$ such that $A \simeq k[\mathbf{A}^n]$ and $k[X]$ is finite over A. In view of the discussion in the previous section, this may be stated as:

6.2. **Theorem** (Geometric form of Noether Normalization)**.** *If X is an affine variety, then there exists a surjective finite morphism $X \to \mathbf{A}^n$.*

6.3. *Remark.* For a brief discussion on the connection with Riemann surfaces, see Ch. 10 §8 in Artin's 'Algebra'.

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