MODULES: THE BASICS

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1. Definitions and basic constructions

1.1. Let A be a ring (commutative with 1). An A-*module* is an abelian group M (written additively) on which A acts linearly. More precisely, it is a pair (M, μ), where A is an abelian group and μ is a map $A \times M \to M$ such that, if we write ax for $\mu(a, x)$ $(a \in A, x \in M)$, the following axioms are satisfied:

(i) $a(x + y) = ax + ay$;
(ii) $(a + a')x = ax + a'x$;
(iii) $(aa')x = a(a'x)$;
(iv) $1x = x$

for all $a, a' \in A$ and all $x, y \in M$.

1.2. *Example.* If A is a field k, then A-module = k-vector space.

1.3. *Example.* A $\mathbf{Z}$-module is the same thing as an abelian group.

1.4. *Example.* An ideal $\mathfrak{a}$ of A is an A-module. In particular, A itself is an A-module.

1.5. *Example.* Let $A = k[x]$ where k is a field. Then an A-module is a k-vector space M with a linear transformation $M \to M$.

1.6. *Example.* The trivial group is an A-module (there is only one possible action). It is denoted by 0.

1.7. Let M, N be A-modules. A map $f\colon M \to N$ is an A-*module homomorphism* (or A-*linear*) if:

(i) $f(x + y) = f(x) + f(y)$;
(ii) $f(ax) = af(x)$

for all $a \in A$ and all $x, y \in M$. The composition of A-module homomorphisms is again an A-module homomorphism.

1.8. *Example.* If A is a field, then an A-module homomorphism is the same thing as a linear transformation of vector spaces.

1.9. *Example.* A $\mathbf{Z}$-module homomorphism is the same thing as a homomorphism of abelian groups.

1.10. An A-module homomorphism $f\colon M \to N$ is an *isomorphism* (often denoted $f\colon M \xrightarrow{\sim} N$) if there exists an A-module homomorphism $f^{-1}\colon N \to M$ such that $f \circ f^{-1}$ and $f^{-1} \circ f$ are the identity map on N and M respectively.

1.11. *Remark.* If it is clear that I am talking about A-modules I will often abbreviate 'A-module homomorphism' to 'morphism'. Further, $M \simeq N$ will denote that M and N are isomorphic.

1.12. Let M, N be A-modules. Then the set of all A-module homomorphisms $M \to N$ can be turned into an A-module as follows: define $f + g$ and af by the rules

$$(f+g)(x) = f(x) + g(x), \quad (af)(x) = af(x)$$

for all $x \in M$ and $a \in A$. This A-module is denoted $\mathrm{Hom}_A(M, N)$ or just $\mathrm{Hom}(M, N)$ (if there is no ambiguity about the ring A). Morphisms $u\colon M' \to M$ and $v\colon N \to N''$ induce maps

$$u^*\colon \mathrm{Hom}(M,N) \to \mathrm{Hom}(M',N) \quad \text{and} \quad v_*\colon \mathrm{Hom}(M,N) \to \mathrm{Hom}(M,N'')$$

defined as follows:

$$u^*(f) = f \circ u, \quad v_*(f) = v \circ f.$$

The maps u^* and v_* are A-module homomorphisms. For any A-module M there is a natural isomorphism $\mathrm{Hom}(A, M) \simeq M$: any A-module homomorphism $f\colon A \to M$ is uniquely determined by $f(1)$, which can be any element of M.

1.13. Let M be an A-module. A *submodule* M' of M is a subgroup of M which is closed under multiplication by elements of A. The abelian group M/M' then inherits an A-module structure from M, defined by $a(x + M') = ax + M'$. The A-module M/M' is the *quotient* of M by M'. If $f\colon M \to N$ is an A-module homomorphism, then the *kernel* of f is the set

$$\ker(f) = \{x \in M \mid f(x) = 0\}$$

and is a submodule of M. If $\ker(f) = 0$, then f is *injective*. The *image* of f is the set

$$\mathrm{im}(f) = f(M)$$

and is a submodule of N. If $\mathrm{im}(f) = N$, then f is *surjective*. The *cokernel* of f is

$$\mathrm{coker}(f) = N/\mathrm{im}(f)$$

which is a quotient module of N. A morphism that is both injective and surjective is an isomorphism:

1.14. **Proposition** (First isomorphism theorem). *Let $f\colon M \to N$ be an A-module homomorphism. Then f induces an isomorphism*

$$\mathrm{im}(f) \simeq M/\ker(f).$$

Proof. Exercise! □

1.15. Let M be an A module and let $\{M_i\}_{i \in I}$ be a family of submodules of M. Their *sum* $\sum M_i$ is the set of all finite sums $\sum x_i$, where $x_i \in M_i$ for all $i \in I$ and almost all the x_i are 0. The set $\sum M_i$ is a submodule of M. It is the smallest submodule of M which contains all the M_i. The *intersection* $\bigcap M_i$ is also a submodule of M.

1.16. **Proposition** (Second isomorphism theorem). *If M_1, M_2 are submodules of M, then*

$$(M_1 + M_2)/M_1 \simeq M_2/(M_1 \cap M_2).$$

Proof. Exercise! □

1.17. **Proposition** (Third isomorphism theorem). *If $N \subseteq M \subseteq L$ are A-modules, then*

$$(L/N)/(M/N) \simeq L/M.$$

Proof. Exercise! □

1.18. In general we cannot 'multiply' two submodules, but we can define $\mathfrak{a}M$, where $\mathfrak{a}$ is an ideal of A and M is an A-module; it is the set of all finite sums $\sum a_i x_i$ with $a_i \in \mathfrak{a}$, $x_i \in M$, and is a submodule of M.

1.19. Let M be an A-module. The *annihilator* of M is

$$\mathrm{Ann}(M) = \{a \in A \mid ax = 0 \text{ for all } x \in M\}.$$

This is an ideal of A. Moreover, if $\mathfrak{a} \subseteq \mathrm{Ann}(M)$ is a sub-ideal, then we may regard M as an $A/\mathfrak{a}$-module as follows: if $\overline{a} \in A/\mathfrak{a}$ is represented by $a \in A$, define $\overline{a}x$ to be ax for all $x \in M$. This is independent of the choice of representative a, since $\mathfrak{a}M = 0$.

1.20. If M, N are A-modules, their *direct sum* $M \oplus N$ is the set of all pairs (x, y) with $x \in M$, $y \in N$. This is an A-module with addition and multiplication defined by:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \qquad a(x, y) = (ax, ay).$$

More generally, if $\{M_i\}_{i\in I}$ is a family of A-modules, we define their direct sum $\bigoplus_{i\in I} M_i$ as follows: its elements are families $(x_i)_{i\in I}$ such that $x_i \in M_i$ for each $i \in I$ and almost all the x_i are 0. If we drop the restriction on the number of non-zero x_i's, then we obtain the *direct product* $\prod_{i\in I} M_i$. Direct sum and direct product are the same if the index set I is finite (but not otherwise, in general).

1.21. **Proposition.** *Let M, N be submodules of L. If $M + N = L$ and $M \cap N = 0$, then $L \simeq M \oplus N$.*

Proof. Define $f\colon M \oplus N \to L, (m, n) \mapsto m + n$. As $M + N = L$, f is surjective. If $f(m + n) = m + n = 0$, then $m = -n$. Consequently, both m, n are in $M \cap N$. So, $m = n = 0$. Hence, f is injective. □

2. Exact sequences

2.1. A sequence of A-modules and A-module homomorphisms

$$\cdots \xrightarrow{f_{i-1}} M^i \xrightarrow{f_i} M^{i+1} \xrightarrow{f_{i+1}} \cdots$$

is said to be *exact at M^i* if $\mathrm{im}(f_{i-1}) = \ker(f_i)$. The sequence is *exact* if it is exact at each M_i.

2.2. *Example.* $0 \to M' \xrightarrow{f} M$ is exact if and only if f is injective.

2.3. *Example.* $M \xrightarrow{g} M'' \to 0$ is exact if and only if g is surjective.

2.4. *Example.* A sequence

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0 \tag{2.4.1}$$

is exact if and only if f is injective, g is surjective and g induces an isomorphism of $\mathrm{coker}(f) = M/\mathrm{im}(f)$ onto M''.

2.5. *Remark.* An exact sequence of type (2.4.1) is often called a *short exact sequence.*

The proof of the following result is not particularly enlightening, however it is a good exercise in keeping many of the definitions so far straight.

2.6. **Proposition.** *Let*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M''$$

be a sequence of A-modules and morphisms. Then this sequence is exact if and only if for all A-modules N, the sequence

$$0 \to \mathrm{Hom}(N, M') \xrightarrow{f_*} \mathrm{Hom}(N, M) \xrightarrow{g_*} \mathrm{Hom}(N, M'')$$

is exact.

Proof. First, assume $0 \to M' \xrightarrow{f} M \xrightarrow{g} M''$ is exact. Let's show exactness at $\mathrm{Hom}(N, M')$. Let $u \in \ker(f_*)$, i.e., $f \circ u = 0$. As f is injective, this implies $u = 0$. That is, we have exactness at $\mathrm{Hom}(N, M')$. Let's show exactness at $\mathrm{Hom}(N, M)$. Let $v \in \mathrm{im}(f_*)$, i.e., $v = f \circ u'$ for some $u' \in \mathrm{Hom}(N, M')$. Then $g_*(v) = g \circ v = g \circ f \circ u' = 0$, since $\mathrm{im}(f) = \ker(g)$. Thus, $\mathrm{im}(f_*) \subseteq \ker(g_*)$. On the other hand, if $v' \in \ker(g_*)$, then $v'(y) \in \ker(g)$ for all $y \in N$. As $\mathrm{im}(f) = \ker(g)$ and f is injective, for each $y \in N$ there is a unique $h(y) \in M'$ such that $v'(y) = f(h(y))$. The map $N \to M', y \mapsto h(y)$ is an A-module homomorphism. Indeed, for $a \in A$,

$$v'(ay) = av'(y) = af(h(y)) = f(ah(y)).$$

By the uniqueness of $h(ay)$ we must have that $h(ay) = ah(y)$. Similarly, if y' is another element in N, then

$$v'(y + y') = v'(y) + v'(y') = f(h(y)) + f(h(y')) = f(h(y) + h(y')).$$

And by the uniqueness of $h(y + y')$, we have $h(y + y') = h(y) + h(y')$. Now $v' = f \circ h = f_*(h)$, i.e., $\ker(g_*) \subseteq \mathrm{im}(f_*)$. Hence, we have exactness at $\mathrm{Hom}(N, M)$.

Now assume $0 \to \mathrm{Hom}(N, M') \xrightarrow{f_*} \mathrm{Hom}(N, M) \xrightarrow{g_*} \mathrm{Hom}(N, M'')$ is exact for all A-modules N. Let's show $0 \to M' \xrightarrow{f} M \xrightarrow{g} M''$ is exact. For exactness at M', take $N = \ker(f)$ and let $i\colon \ker(f) \to M'$ be the inclusion map. Then $f_*(i) = f \circ i = 0$. As f_* is injective this implies $\ker(f) = 0$. Let's show exactness at M. Take $N = M$ and let id_M be the identity map on M. Then $g \circ f = g_*(f) = g_* f_*(\mathrm{id}_M) = 0$, by exactness at $\mathrm{Hom}(N, M) = \mathrm{Hom}(M, M)$. Hence, $\mathrm{im}(f) \subseteq \ker(g)$. Now take $N = \ker(g)$ and let $j\colon \ker(g) \to M$ be the inclusion map. Then $j \in \ker(g_*)$, so $j = \mathrm{im}(f_*)$, by exactness at $\mathrm{Hom}(N, M) = \mathrm{Hom}(\ker(g), M)$. That is, $j = f \circ k$ for some $k \in \mathrm{Hom}(\ker(g), M')$. Consequently, $\ker(g) \subseteq \mathrm{im}(f)$. □

2.7. *Example.* If $N' \xrightarrow{f} N \xrightarrow{g} N'' \to 0$ is exact, then it is not necessarily true that $\mathrm{Hom}(M, N') \xrightarrow{f_*} \mathrm{Hom}(M, N) \xrightarrow{g_*} \mathrm{Hom}(M, N'') \to 0$ is exact. For instance, consider the exact sequence of $\mathbf{Z}$-modules $\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0$ where $\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$ is the obvious quotient map. Then clearly $\mathrm{Hom}(\mathbf{Z}/2\mathbf{Z}, \mathbf{Z}) \to \mathrm{Hom}(\mathbf{Z}/2\mathbf{Z}, \mathbf{Z}/2\mathbf{Z}) \to 0$ is not exact.

2.8. **Proposition.** *Let*

$$N' \xrightarrow{f} N \xrightarrow{g} N'' \to 0$$

be a sequence of A-modules and A-module homomorphisms. Then this sequence is exact if and only if for all A-modules M, the sequence

$$0 \to \mathrm{Hom}(N'', M) \xrightarrow{g^*} \mathrm{Hom}(N, M) \xrightarrow{f^*} \mathrm{Hom}(N', M)$$

is exact.

Proof. Exercise! □

3. Free modules

3.1. A *free* A-module is one which is isomorphic to an A-module of the form $\bigoplus_{i\in I} M_i$, where each $M_i \simeq A$ (as an A-module). A free module that is isomorphic to $A \oplus \cdots \oplus A$ (n summands) is said to have *rank* n. The module $A \oplus \cdots \oplus A$ (n-summands) is often denoted by $A^{\oplus n}$ or A^n. By convention $A^{\oplus 0}$ is the zero module. The notion of rank is well defined:

3.2. **Proposition.** *If $A^{\oplus n} \simeq A^{\oplus m}$, then $m = n$.*

Proof. Exercise! Hint: let $\mathfrak{m}$ be a maximal ideal of A and consider $A/\mathfrak{m}$. □

3.3. *Example.* A submodule of a free module need not be free. Even a direct summand of a free module need not be free (M is a direct summand of L if $L \simeq M \oplus N$ for some module N). For instance, let $A = \mathbf{Z}/6\mathbf{Z}$, then as an A-module $\mathbf{Z}/6\mathbf{Z} \simeq \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z}$. However, neither $\mathbf{Z}/2\mathbf{Z}$ nor $\mathbf{Z}/3\mathbf{Z}$ are free modules.

It is slightly harder to construct an example as above if we require A to be an integral domain. Such an example is outlined in the problem set.

3.4. **Proposition.** *Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be an exact sequence of A-modules. Let N be a free module. Then*

$$0 \to \mathrm{Hom}(N, M') \xrightarrow{f_*} \mathrm{Hom}(N, M) \xrightarrow{g_*} \mathrm{Hom}(N, M'') \to 0$$

is exact.

Proof. Prop. 2.6 gives exactness at $\mathrm{Hom}(N, M')$ and $\mathrm{Hom}(N, M)$. By definition, $N \simeq \bigoplus_{i\in I} A_i$. For each $j \in I$, let e_j denote the image of $1 \in A_j \subseteq \bigoplus_{i\in I} A_i$ under this isomorphism. Let $u \in \mathrm{Hom}(N, M'')$. For each $i \in I$, pick $x_i \in M$ such that $g(x_i) = u(e_i)$. Define $v \in \mathrm{Hom}(N, M)$ by $v\colon e_i \mapsto x_i$ for all $i \in I$. Then $g_*(v) = u$. □

3.5. **Corollary.** *Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be an exact sequence. If M'' is free, then $M \simeq M' \oplus M''$.*

Proof. Exercise! □

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