PROBLEM SET 1

DUE APRIL 5

All rings are commutative with 1.

1. Regular problems

1.1. Let M be an A-module and $M' \subseteq M$ a submodule. Prove that the additive inverse of an element of M' is in M'.

1.2. Verify the following assertions made in lecture:

(i) The composition of A-module homomorphisms is again an A-module homomorphism.
(ii) Let M, N be A-modules. Then $\operatorname{Hom}_A(M, N)$ is an A-module (with multiplication as defined in class, see the notes).
(iii) Let $f\colon M \to N$ be an A-module homomorphism. Then $\ker(f)$ and $\operatorname{im}(f)$ are submodules of M and N respectively.
(iv) Let M be an A-module. Then $\operatorname{Ann}(M)$ is an ideal of A.

1.3.

(i) Compute $\operatorname{Hom}_{\mathbf{Z}}(\mathbf{Z}/12\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}, \mathbf{Z}/30\mathbf{Z})$.
(ii) What is the annihilator of the $\mathbf{Z}$-module $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$? What is the annihilator of the $\mathbf{Z}$-module $\mathbf{Z}$?

1.4.

(i) Let M be an abelian group. Prove that if M has a structure of a $\mathbf{Q}$-module with its given law of composition as addition, then this structure is uniquely determined.
(ii) Prove that no finite abelian group has a $\mathbf{Q}$-module structure.

1.5. An A-module is called *simple* (or *irreducible*) if it is non-zero and contains no proper non-zero submodules.

(i) Prove that any simple A-module L is isomorphic to $A/\mathfrak{m}$ where $\mathfrak{m}$ is a maximal ideal.
(ii) Prove *Schur's lemma*: let $f\colon L \to L'$ be a morphism of simple modules. Then either f is 0 or an isomorphism.
(iii) Let L be an A-module. Set $\operatorname{End}_A(L) = \operatorname{Hom}_A(L, L)$. Then $\operatorname{End}_A(L)$ is ring via composition of maps. Prove that if L is simple, then $\operatorname{End}_A(L)$ is a field.

1.6. If $A^{\oplus m} \simeq A^{\oplus n}$, then show that $m = n$. Hint: let $\mathfrak{m} \subset A$ be a maximal ideal and consider $A/\mathfrak{m}$.

1.7.

(i) Let A be a ring, $\mathfrak{a} \subseteq A$ an ideal. Prove or find a counterexample: if $A/\mathfrak{a}$ is a free A-module, then $\mathfrak{a} = 0$.
(ii) Let A be a ring such that every finitely generated A-module is free. Show that A is either a field or the zero ring.

1.8. Let $A = \mathbf{C}[x, y]$ and let $\mathfrak{a}$ be the ideal of A generated by the elements x and y. Is $\mathfrak{a}$ a free A-module? Prove or disprove.

1.9. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence. Prove that if M'' is free, then $M \simeq M' \oplus M''$.

2. Optional problems

2.1. Prove the First, Second and Third isomorphism theorems for modules (see the notes if you don't remember what these are).

2.2. Let

$$N' \xrightarrow{f} N \xrightarrow{g} N'' \to 0$$

be a sequence of A-modules and A-module homomorphisms. Prove that this sequence is exact if and only if for all A-modules M, the sequence

$$0 \to \mathrm{Hom}(N'', M) \xrightarrow{g^*} \mathrm{Hom}(N, M) \xrightarrow{f^*} \mathrm{Hom}(N', M)$$

is exact.

2.3. Let $A = \mathbf{Z}[\sqrt{-5}]$. Let $\mathfrak{p}_1$ be the ideal generated by 3 and $1 + \sqrt{-5}$, let $\mathfrak{p}_2$ be the ideal generated by 3 and $1 - \sqrt{-5}$.

- (i) Show that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are maximal ideals.
- (ii) Show that $\mathfrak{p}_1 + \mathfrak{p}_2 = A$.
- (iii) Show that $\mathfrak{p}_1 \cap \mathfrak{p}_2 = \mathfrak{p}_1\mathfrak{p}_2$.
- (iv) Show that $\mathfrak{p}_1\mathfrak{p}_2$ is the ideal generated by 3. Conclude that $\mathfrak{p}_1\mathfrak{p}_2 \simeq A$ as an A-module.
- (v) Let $f\colon \mathfrak{p}_1 \oplus \mathfrak{p}_2 \to A$ be the morphism given by $(x, y) \mapsto x + y$. Show that $\ker(f) \simeq \mathfrak{p}_1 \cap \mathfrak{p}_2 = \mathfrak{p}_1\mathfrak{p}_2$.
- (vi) Show that neither $\mathfrak{p}_1$ nor $\mathfrak{p}_2$ are principal ideals.
- (vii) Conclude that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are (isomorphic) direct summands of a free A-module but are not themselves free. Hint: construct an exact sequence $0 \to A \to \mathfrak{p}_1 \oplus \mathfrak{p}_2 \to A \to 0$.

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