PROBLEM SET 2

DUE APRIL 12

All rings are commutative with 1.

1. Regular problems

1.1. Show that a basis for a module is necessarily a minimal set of generators. Is the converse true?

1.2. Let A be a ring and let M be a free A-module of finite rank. Prove or find a counterexample to the following:

(i) Every set of generators of M contains a basis.
(ii) Every linearly independent set of elements of M can be extended to a basis.

1.3. Let M be a finitely generated A-module. Suppose that $\mathfrak{m}M = M$ for all maximal ideals $\mathfrak{m} \subset A$. Show that $M = 0$. Hint: take $\mathfrak{m}$ to be a maximal ideal containing $\mathrm{Ann}(M)$ (is it gauranteed that such an $\mathfrak{m}$ exists?). Then apply Nakayama's lemma to get a special element in A. Where does this special element live?

1.4. Let

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence of A-modules. Show that:

(i) if M' and M'' are finitely generated, then so is M;
(ii) if M is finitely generated, then so is M''.

1.5. Find an A-module M and a submodule $N \subset M$ such that M is finitely generated but N is not. Hint: look at the next problem.

1.6. Let A be a noetherian ring. Prove that every submodule of a finitely generated A-module is finitely generated. Hint: this is proved in almost every algebra textbook, including Artin.

1.7. Prove that the following rings are noetherian:

(i) $\mathbf{Z}$;
(ii) $\mathbf{C}[x]$;

Hint/comment: if you decide to state that each of these rings is a PID and hence noetherian, then you need to prove that these rings are indeed PIDs.

1.8. Put the following integer matrices in Smith normal form by integer row and column operations.

(i) $\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$
(ii) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$
(iii) $\begin{pmatrix} 3 & 1 & -4 \\ 2 & -3 & 1 \\ -4 & 6 & -2 \end{pmatrix}$

2. Optional problems

2.1. Prove the *Hilbert basis theorem*: If A is a noetherian ring, then so is $A[x]$.

2.2. Show directly (i.e., without appealing to the Hilbert basis theorem) that $\mathbf{Z}[x]$ is noetherian.

2.3. Recall that a *local ring* is a ring A with a unique maximal ideal $\mathfrak{m}$. Let A be a local ring. Show that a direct summand of a free A-module of finite rank is free.

2.4. Formulate and prove a uniqueness statement about Smith normal form over $\mathbf{Z}$. What about over an arbitrary PID?

2.5. Recall that a complex number $\alpha \in \mathbf{C}$ is called an *algebraic integer* if it is the zero of some monic polynomial with integer coefficients, i.e.,

$$\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0$$

for some $a_0, \ldots, a_{n-1} \in \mathbf{Z}$.

(i) Let α be a complex number and let $\mathbf{Z}[\alpha]$ be the subring of $\mathbf{C}$ generated by α. Prove that α is an algebraic integer if and only if $\mathbf{Z}[\alpha]$ is a finitely generated abelian group. Hint: it may help to look at the proof of Nakayama's lemma.

(ii) Prove that if α, β are algebraic integers, then the subring $\mathbf{Z}[\alpha, \beta]$ of $\mathbf{C}$ which they generate is a finitely generated abelian group.

(iii) Prove that the algebraic integers form a subring of $\mathbf{C}$.

Department of Mathematics, University of California, Davis, CA 95616
E-mail address: `virk@math.ucdavis.edu`