PROBLEM SET 3

DUE APRIL 19

1. Regular problems

1.1. Let F be an $n \times n$ matrix with coefficients in some commutative ring (with 1) A. Let $f\colon A^{\oplus n} \to A^{\oplus n}$ be left multiplication by F (in terms of the 'usual' basis for $A^{\oplus n}$) and let d denote the ideal generated by $\det F$. Prove or find a counterexample: the image of f is equal to dA^n.

1.2. Give an example of the following (remember you must prove/justify why your example satisfies the required conditions):

(i) a module that is *not* finitely generated;
(ii) a module that is finitely generated but *not* finitely presented.

1.3. Write the abelian group generated by x, y and the relation $3x + 4y = 0$ as a direct sum of cyclic groups.

1.4. Let $i = \sqrt{-1}$ and let V be the $\mathbf{Z}[i]$ module generated by elements v_1, v_2 and relations

$$(1+i)v_1 + (2-i)v_2 = 0, \qquad 3v_1 + 5iv_2 = 0.$$

Write this module as a direct sum of cyclic modules.

1.5. Classify finitely generated modules over each of the following rings:

(i) $\mathbf{Z}/4\mathbf{Z}$;
(ii) $\mathbf{Z}/6\mathbf{Z}$;
(iii) $\mathbf{Z}/n\mathbf{Z}$.

1.6. Let $K \supset F$ be a field extension, $\alpha \in K$ algebraic over F. Let $f(x)$ be the irreducible polynomial of α over F. Show that this terminology is justified, i.e., $f(x)$ is indeed irreducible over F.

1.7. Show that $\mathbf{Q}(\sqrt{-1}, \sqrt{5}) = \mathbf{Q}(\sqrt{-1} + \sqrt{5})$.

1.8. Let α be the real cube root of 2. Compute the irreducible polynomial for $1 + \alpha^2$ over $\mathbf{Q}$.

1.9. Determine the irreducible polynomial for $\alpha = \sqrt{3} + \sqrt{5}$ over each of the following fields.

(i) $\mathbf{Q}$;
(ii) $\mathbf{Q}(\sqrt{5})$;
(iii) $\mathbf{Q}(\sqrt{10})$;
(iv) $\mathbf{Q}(\sqrt{15})$.

2. Optional problems

2.1. Let G be a finite abelian group and let $\chi\colon G \to \mathbf{C}^\times$ be a non-trivial homomomorphism. Prove that

$$\sum_{g \in G} \chi(g) = 0.$$

2.2. An abelian group G is said to be *torsion free* if $g \in G$, $g \neq 0$ and $kg = 0$ for $k \in \mathbf{Z}$ implies $k = 0$. Prove that a finitely generated torsion free abelian group is a finitely generated free group.

2.3. Let A be a subring of $\mathbf{C}[x]$ that contains $\mathbf{C}$ but is not equal to $\mathbf{C}$. Show that $\mathbf{C}[x]$ is a finitely generated A-module.

2.4. Let $\varphi \colon \mathbf{Z}^n \to \mathbf{Z}^n$ be a homomorphism given by multiplication by an integer matrix A. Show that the image of φ is of finite index if and only if A is non-singular and that if so, then the index is equal to $|\det A|$.

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