PROBLEM SET 4

DUE APRIL 26

1. Regular problems

1.1. Prove that if $F(\alpha)$ is a simple algebraic extension of a field F, then $[F(\alpha) : F]$ is the degree of the irreducible polynomial of α over F.

1.2. Prove that a simple extension $F(\alpha) \supset F$ is algebraic over F if and only if $[F(\alpha) : F]$ is finite.

1.3. Let $F \subset K \subset L$ be fields. Prove that if K is algebraic over F and L is algebraic over K, then L is algebraic over F.

1.4. Prove or find a counterexample: every algebraic extension is a finite extension.

1.5. Let $F(\alpha)$ be a simple extension of F with the property that $[F(\alpha) : F] = 13$. Prove or find a counterexample: $F(\alpha^5) = F(\alpha)$.

1.6. Let F be a field of characteristic $p \neq 0$. Show that $\{0, 1, \ldots, p-1\}$ is a sub-field of F isomorphic to $\mathbf{F}_p$.

1.7. Prove or find a counterexample: every field of non-zero characteristic is finite.

2. Optional problems

2.1. Let $K \supset \mathbf{C}$ be a field extension of the complex numbers. Show that if the degree $[K : \mathbf{C}]$ is finite, then $K = \mathbf{C}$.

2.2. Let K be a field of characteristic $p \neq 0$. The *Frobenius map* $\mathrm{Fr}\colon K \to K$ is defined by $x \mapsto x^p$. Show that Fr is (i) a ring homomorphism (ii) injective.

2.3. Retain the notation of the previous problem. Recall that $\mathbf{F}_p$ is a sub-field of K. Show that Fr restricted to $\mathbf{F}_p$ is the identity.

2.4. Retain the notation of the previous two problems. For a positive integer n let $\mathrm{Fr}^n\colon K \to K$ denote the ring homomorphism obtained by applying Fr n-times. Set

$$K^{\mathrm{Fr}^n} = \{x \in K \mid \mathrm{Fr}^n(x) = x\}.$$

Show that K^{Fr^n} is a finite sub-field of K.

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