PROBLEM SET 7

DUE MAY 17

1. Regular problems

1.1. Show that

(i) $\text{zeroes}(0) = k^n$ and $\text{zeroes}(1) = \emptyset$.

(ii) For any family of ideals $\mathfrak{a}_i \in k[x_1, \ldots, x_n]$, $i \in I$:

$$\text{zeroes}(\bigcup_{i \in I} \mathfrak{a}_i) = \bigcap_{i \in I} \text{zeroes}(\mathfrak{a}_i).$$

(iii) $\text{zeroes}(\mathfrak{a} \cap \mathfrak{b}) = \text{zeroes}(\mathfrak{a}\mathfrak{b}) = \text{zeroes}(\mathfrak{a}) \cup \text{zeroes}(\mathfrak{b})$ for any ideals $\mathfrak{a}, \mathfrak{b} \subseteq k[x_1, \ldots, x_n]$.

(iv) Let $\mathfrak{a}, \mathfrak{b} \subseteq k[x_1, \ldots, x_n]$ be ideals. If $\mathfrak{a} \subseteq \mathfrak{b}$, then $\text{zeroes}(\mathfrak{b}) \subseteq \text{zeroes}(\mathfrak{a})$.

(v) Let $X, Y \subseteq k^n$ be algebraic sets. If $X \subseteq Y$, then $I(Y) \subseteq I(X)$.

(vi) $\text{zeroes}(I(X)) = X$ for all algebraic sets $X \subseteq k^n$.

(vii) $\sqrt{\mathfrak{a}} \subseteq I(\text{zeroes}(\mathfrak{a}))$ for all ideals $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$.

1.2. Let k be a field. Prove or find a counterexample: every algebraic subset of k consists of finitely many points.

1.3. Show that a morphism of algebraic sets $f \colon X \to Y$ is an isomorphism if and only if $f^* \colon k[Y] \to k[X]$ is an isomorphism.

1.4. Prove or find a counterexample: there exists no algebraic set X with $k[X] \simeq k[z]/z^2$.

1.5. Let $C \subset k^2$ be the set of solutions of the polynomial $x^3 - y^2 = 0$. Define $f \colon \mathbf{A}^1 \to C$ by $t \mapsto (t^2, t^3)$. Show that this morphism is not an isomorphism of algebraic sets. Sketch $x^3 - y^2 = 0$ in $\mathbf{R}^2$. Make an observation regarding your sketch and the fact that f is not an isomorphism.

1.6. Let B be an integral domain and let $A \subseteq B$ be a subring. Assume that B is integral over A. Show that A is a field if and only if B is a field.

1.7. Let k be a field and let $B = k[x, y]/(x^3 - y^2)$. Find a k-subalgebra $A \subseteq B$ such that B is finite over A and A is isomorphic to a polynomial ring over k. Sketch $x^3 - y^2 = 0$ in $\mathbf{R}^2$ and interpret your construction in terms of this picture.

1.8. Let k be a field. Let $g \in k[x_1, \ldots, x_n]$. Show that the map

$$f \colon k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n, t]/(1 - tg), \quad h(x_1, \ldots, x_n) \mapsto h(x_1, \ldots, x_n)$$

is injective whenever $g \neq 0$.

2. Optional problems

2.1. Let k be a field and let $B = k[x, y]/(xy - 1)$. Find a k-subalgebra $A \subseteq B$ such that B is finite over A and A is isomorphic to a polynomial ring over k. Sketch $xy - 1 = 0$ in $\mathbf{R}^2$ and interpret your construction in terms of this picture. Note that taking $A = k[x]$ above does not work. Make an observation relating this fact and the picture you just sketched.

2.2. Let $C \subset k^2$ be the set of solutions of the polynomial $x^3 - y^2 = 0$. Show that $k[C] = k[x,y]/(x^3 - y^2)$.

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