PROBLEM SET 8

DUE MAY 24

k will denote an algebraically closed field k. All varieties will be over k.

1. Regular problems

1.1. Let $\phi\colon A \to B$ be a morphism of finitely generated k-algebras. Let $\mathfrak{m} \subseteq B$ be a maximal ideal. Show that $\phi^{-1}(\mathfrak{m})$ is a maximal ideal in A.

1.2. Find commutative rings A, B, a ring homomorphism $\phi\colon A \to B$ and a maximal ideal $\mathfrak{m} \subseteq B$ such that $\phi^{-1}(\mathfrak{m})$ is not maximal in A.

1.3. Let $f\colon X \to Y$ be a morphism of affine varieties and let $Z \subseteq Y$ be closed. Show that

$$f^{-1}(Z) = \text{zeroes}(f^*(I(Z))).$$

1.4. Prove or find a counterexample: if $f\colon X \to Y$ is a morphism of affine varieties and $Z \subseteq X$ is a closed subset, then $f(Z)$ is closed in Y.

1.5. Let X be an affine variety and let $Z \subseteq X$ be a closed subset. Set $k[Z] = i^*(k[X])$, where $i\colon Z \hookrightarrow X$ is the inclusion map. Show that $k[Z] \simeq k[X]/I(Z)$ and that this defines the structure of an affine variety on Z.

1.6. Show that for a commutative ring A the following definitions of a prime ideal $\mathfrak{p} \subseteq A$ are equivalent:

(i) If $xy \in \mathfrak{p}$, then either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.
(ii) If $\mathfrak{a}, \mathfrak{b} \subseteq A$ are ideals such that $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$, then either $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$.

1.7. Let X be an affine variety and let $Z \subseteq X$ be an irreducible closed subset. Show that Z is contained in some component of X. In other words, the components of X are the maximal irreducible closed subsets of X.

1.8. Let X, Y be sets and let $f\colon X \to Y$ be a map. Let $k[X], k[Y]$ be the rings of all functions $X \to k$ and $Y \to k$ respectively (multiplication and addition in these rings defined pointwise). Then we have a ring homomorphism $f^*\colon k[Y] \to k[X]$ given by $f^*p(x) = p(f(x))$. Show that if f is surjective, then f^* is injective. Find a counterexample showing that the converse is false.

1.9. Let B be a commutative ring and $A \subseteq B$ a subring. Let $\mathfrak{m}$ be a maximal ideal in A. Show that if B is finite over A, then there exists a maximal ideal $\mathfrak{m}'$ of B such that $\mathfrak{m} = \mathfrak{m}' \cap A$.

1.10. Let A be a k-algebra that is finite dimensional as a k-vector space. Show that A has only finitely many maximal ideals.

2. Optional problems

2.1. Let X be an affine variety. Show that a closed subset $Z \subseteq X$ is an irreducible component if and only if $I(Z)$ is a minimal prime ideal in $k[X]$.

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