Math 748 Homework 4

Due Wednesday, October 4

1. Show the ring $\mathbb{Z}[x]$ is Noetherian and integrally closed in its field of fractions, but is not a Dedekind domain.

2. Let R be a subring of the ring of integers O_K of a number field K. Show that the following are equivalent:

   (a) The index $[O_K : R]$ (as abelian groups) is finite.

   (b) R contains a basis of K over $\mathbb{Q}$.

   (c) The field of fractions of R is K.

   Rings satisfying these conditions are called *orders* in K. Show that every order R in K satisfies:

   (i) R is Noetherian

   (ii) Every prime ideal of R is maximal.

   (iii) If $R \neq O_K$ then R is not integrally closed in K.

3. Give an example that shows that unique factorization of ideals fails in the ring $\mathbb{Z}[\sqrt{-3}]$. (Hint: let $\mathfrak{a} = (2, 1 + \sqrt{-3})$ and show $\mathfrak{a} \neq (2)$ but $\mathfrak{a}^2 = (2)\mathfrak{a}$. Conclude unique factorization fails.)

4. Let B be an integral domain in which all nonzero ideals admit a unique factorization into prime ideals. Show that B is a Dedekind domain.