Math 748 Homework 6

Due Wednesday, October 18

1. Let $A \subseteq B$ be an extension of Dedekind domains, and let $\mathfrak{p}$ be a prime ideal in A. Show that $\mathfrak{p}B \neq B$. (This implies there is at least one prime of B lying over $\mathfrak{p}$.)

2. Find all the primes that ramify in the (ring of integers of) the number field $\mathbb{Q}(\sqrt{-63})$. Now do the same for $\mathbb{Q}(\sqrt{-57})$. Try to do this without looking at Milne p. 54. Feel free to use any past homework problem about quadratic number fields.

For the next three problems, you may use the computer algebra system of your choice to compute *polynomial discriminants, minimal polynomials of elements in number fields, and factorizations of polynomials with coefficients in finite fields.* Please do all other computations by hand, including discriminants of any bases not consisting of powers of a single element. As for computer algebra systems, my personal favorite is Magma, for which we have a departmental license (so it's available free to all: just ssh magma1 and then type "magma" at the prompt). If you would like some advice on getting started with Magma, please contact me.

3. Find all the primes that ramify in the (ring of integers of) the number field $\mathbb{Q}(\sqrt[3]{10})$. Be sure to prove that all the primes you list do indeed ramify.

4. Find an element α such that the ring of integers O_K of $\mathbb{Q}(\sqrt[3]{10})$ is $\mathbb{Z}[\alpha]$ (problem 3 may help you decide what kind of element to look for). Using the fact that $O_K = \mathbb{Z}[\alpha]$, find the factorizations (including generators for the prime ideal factors) of the ideals $2O_K, 3O_K, 7O_K, 11O_K, 37O_K$.

5. Find all primes that ramify in the ring of integers O_K for $K = \mathbb{Q}(\sqrt{-1}, \sqrt{5})$. Hints: $\frac{1+\sqrt{5}}{2}$ is in O_K and $\mathbb{Q}(\sqrt{-5}) \subset K$.