Math 748 Homework 8

Due Monday, October 30

1. Let $K = \mathbb{Q}(\alpha)$, where α is a root of $x^3 - 3$, and take for granted the fact that $O_K = \mathbb{Z}[\alpha]$. Consider the ideal $\mathfrak{a} = (5, \alpha + 3) = 5\mathbb{Z} + (\alpha + 3)\mathbb{Z} + (\alpha^2 + 1)\mathbb{Z}$. (Note that the second equality is not obvious.) Write down the basis vectors for $\sigma(\mathfrak{a})$ (thought of as a lattice in $\mathbb{R}^3$). By taking the appropriate determinant, find $\mathrm{Vol}(\sigma(\mathfrak{a}))$ (do this by hand – it doesn't take long). Now compute $2^{-s}\mathbb{N}(\mathfrak{a})|\Delta_K|^{1/2}$. (Hint on finding $\mathbb{N}(\mathfrak{a})$: what $p \in \mathbb{Z}$ does $\mathfrak{a}$ lie over? What primes lie over this p?) By our theorem from class, the two answers should agree!

2. Using the norm on $V := \mathbb{R}^r \times \mathbb{C}^s \cong \mathbb{R}^n$ that we defined in class on Monday, 10/23, find by hand the volume of $S(7) := \{\vec{x} \in V : ||\vec{x}|| \leq 7\}$ in the case where $r = 3$ and also $r = s = 1$. Check that this is the same value predicted by the theorem from class on Wednesday 10/25. In the latter case, is the volume of $S(7)$ large enough to ensure that $S(7)$ contains a nonzero point of the lattice $\sigma(\mathfrak{a})$ from the first problem?