Math 748 Homework 9

Due Wednesday, November 8

Note: there are two pages to this assignment

1. In class, we showed that if $K = \mathbb{Q}(\sqrt{-d})$ is an imaginary quadratic field ($d > 0$ squarefree) and a primitive mth root of unity ζ_m is contained in K, then $m = 1, 2, 3, 4$, or 6. Use this to determine for all K the set of roots of unity $\mu(K)$ contained in K.

2. The continued fraction expansion for $\alpha \in \mathbb{R}$ is the writing of α as

$$a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots}}} \tag{1}$$

with each $a_i \in \mathbb{Z}$. To find the a_i, first let $[\alpha]$ be the greatest integer less than or equal to α, so that $a_0 = [\alpha]$. Let β be the reciprocal of the fractional part $\alpha - [\alpha]$, so that from (1) we have $\beta = a_1 + (1/(a_2 + \cdots))$. Thus $a_1 = [\beta]$. Continue in this manner to obtain the other a_i.

If we truncate the expression in (1) at the nth step, we obtain a rational number p_n/q_n. For instance, $p_0/q_0 = a_0/1, p_1/q_1 = a_0 + 1/a_1 = (a_0a_1 + 1)/a_1$. The numbers p_n and q_n are called the *convergents* of α, and are given by the Fibonacci-like recurrences

$$p_{n+1} = a_{n+1}p_n + p_{n-1} \qquad p_{q+1} = a_{n+1}q_n + q_{n-1}$$

with initial values p_0, p_1, q_0, q_1 as given above. The rational numbers p_n/q_n give successively better approximations of α.

Now let $\alpha = \sqrt{d}$, where $d > 0$ is squarefree and $d \equiv 2, 3 \bmod 4$. The numbers p_n/q_n are nearly $\sqrt{d}$, meaning that $p_n^2/q_n^2 - d$ is small. Thus it should not be surprising that $p_n^2 - dq_n^2$ is a small integer. More surprisingly, there is the following result first proved by Lagrange: let $a^2 - db^2 = \pm 1$ for some $a, b \in \mathbb{Z}$. Then $a/b = p_n/q_n$ for some n. Since both p_n and q_n strictly increase with n, it follows that the smallest n with $p_n^2 - dq_n^2 = \pm 1$ gives us the fundamental unit in $\mathbb{Q}(\sqrt{d})$: it's $\epsilon = p_n + q_n\sqrt{d}$ (that's the only one that's greater than 1). Thus we have an algorithm for finding ϵ, namely keep computing convergents until you find p_n/q_n such that $p_n^2 - dq_n^2 = \pm 1$. Since the Unit Theorem tells us there must be a fundamental unit, this must happen for some n. In fact, it is possible to say which one. The continued fraction expansion for $\sqrt{d}$ is eventually periodic (this is true for α if and only if α is contained in a quadratic extension of $\mathbb{Q}$). If the length of this cycle is l, then $n = l - 1$ gives the fundamental unit.

Using this, find the fundamental unit of the ring of integers in $\mathbb{Q}(\sqrt{11}), \mathbb{Q}(\sqrt{19})$, and $\mathbb{Q}(\sqrt{22})$. Don't use a computer, except to perform basic arithmetic to find the appropriate continued fraction expansions and to compute $p_n^2 - dq_n^2$.

3. Let K be a cubic number field with exactly one real embedding, and let ϵ be the unique fundamental unit > 1. We wish to show that $\epsilon^3 > (|\Delta_K| - 24)/4$. In class, we did the following: let $\rho e^{i\theta}, \rho e^{-i\theta}$ be the nonreal conjugates of ϵ. We showed that $\epsilon = \rho^{-2}$. We calculated $|\Delta'|$, where

Δ' is the discriminant of $\{1, \epsilon, \epsilon^2\}$; indeed, we showed that $|\Delta'| = (c - \cos\theta)^2 \cdot 16\sin^2\theta$, where $2c = \rho^3 + \rho^{-3}$. We determined that the largest $|\Delta'|$ can be is when $\cos\theta = \beta$, where β is a root of $x^2 - (c/2)x - 1/2$ and $\rho^6 - 4\beta^2 - 4\beta^4 < 0$. Converting the expression for $|\Delta'|$ into a function of $\cos\theta$ now gives

$$|\Delta'| \leq 16(c^2 - 2\beta c + \beta^2)(1 - \beta^2). \tag{2}$$

Use (2), the fact that $\beta^2 - (c/2)\beta - 1/2 = 0$, and the bound involving ρ and β to deduce that $|\Delta'| < 4\rho^{-6} + 24$. Then use this to conclude that $\epsilon^3 > (|\Delta_K| - 24)/4$.

Bonus Problem (2 points): Use the above style of argument to derive a lower bound for the fundamental unit in a real quadratic field. This bound can sometimes be used to give an alternate proof to Lagrange's that the units produced by the algorithm in problem 2 are indeed fundamental.

4. Find the fundamental unit in $K = \mathbb{Q}(\sqrt[3]{2})$ (Hint: $\Delta_K = -108$. Also, what is the norm of $\sqrt[3]{2} - 1$?)