Solution 1.

(a) Let α, f, r be as stated in the problem. Consider the polynomial $g(x) = f(x+r)$, note that
$$g(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$$
where each $a_i \in \mathbb{Z}$. Observe that $a_n = g(0) = f(0+r) = f(r) = \pm 1$, furthermore $g(\alpha - r) = f(\alpha - r + r) = f(\alpha) = 0$, thus we have that
$$0 = (\alpha - r)^n + a_1(\alpha - r)^{n-1} + \cdots + a_{n-1}(\alpha - r) + a_n$$
which implies that
$$\pm 1 = (\alpha - r)[(\alpha - r)^{n-1} + a_1(\alpha - r)^{n-2} + \cdots + a_{n-1}]$$
hence $\alpha - r$ is a unit in $\mathbb{Z}[\alpha]$.

(b) Note that $\sqrt[3]{7}$ is a root of the monic polynomial $f(x) = x^3 - 7$, furthermore $f(2) = 1$, thus by part (a), $\sqrt[3]{7} - 2$ is a unit in $\mathbb{Z}[\sqrt[3]{7}]$, consequently as $\mathbb{Z}[\sqrt[3]{7}] \subseteq O_K$, we have that $\sqrt[3]{7} - 2$ is a unit in O_K, which implies $-(\sqrt[3]{7} - 2) = 2 - \sqrt[3]{7}$ is also a unit in O_K, which further gives us that $\frac{1}{2-\sqrt[3]{7}}$ is a unit in O_K. Let ϵ be the fundamental unit of O_K, we then know that $\epsilon > \sqrt[3]{\frac{\Delta_K - 24}{4}} = \sqrt[3]{\frac{1323-23}{4}} \approx 6.87$. Now note that $\frac{1}{2-\sqrt[3]{7}} \approx 11.48$, as $\frac{1}{2-\sqrt[3]{7}}$ is a power of ϵ and $\epsilon^2 > 36$, we must have that $\epsilon = \frac{1}{2-\sqrt[3]{7}}$.

(c) Note that $Disc(\mathbb{Z}[\alpha]) = -247 = -(13)(19)$ which is squarefree and thus the ring of integers $O_K = \mathbb{Z}[\alpha]$. Let ϵ be the unique real fundamental unit > 1. We know that $\epsilon > \sqrt[3]{\frac{247-24}{4}} \approx 3.82$. Furthermore, $f(x) = x^3 + x - 3$ is the minimal polynomial of α and $f(1) = -1$, thus by part (a) $\alpha - 1$ is a unit in O_K, this in turn implies that $\frac{1}{\alpha - 1}$ is a unit in O_K. Note that O_K contains all the roots of $x^3 + x - 3$ and we may thus assume that α is a real root (whose existence is assured by the intermediate value theorem). Furthermore, note that (by the intermediate value theorem again) $1.1 < \alpha < 1.5$ thus $2 < \frac{1}{\alpha-1} < 10$ is a power of ϵ, but $\epsilon > 3.82$ implies that $\epsilon^2 > 10$, as $\frac{1}{\alpha-1}$ must be a power of ϵ we must have that $\frac{1}{\alpha-1} = \epsilon$.

□

Solution 2.

Let T denote the number of Harold's troops. Then the conditions of the problem simply state that $T = 13b^2$ and $T + 1 = a^2$ for some $a, b \in \mathbb{N}$, thus we seek solutions to the Pell equation: $a^2 - 13b^2 = 1$. Using the continued fraction method we obtain the smallest solution of $a = 649$ and $b = 180$ (Maple verifies that $649^2 - 13(180)^2 = 1$. Thus, $T = 13(180)^2 = 421200$ (which is a pretty large number for an army in the 1100s).

Remark: I understand that one of the hints for the problem mentions something to the effect of solving $a^2 - 13b^2 = \pm 4$ to find the fundamental unit of O_K (where $K = \mathbb{Q}(\sqrt{13})$). The solution of $a^2 - 13b^2 = \pm 4$ gives us a fundamental unit for O_K because $13 \equiv 1 \mod 4$, we have that $O_K = \mathbb{Z}[\frac{1+\sqrt{13}}{2}]$, setting the norm of an arbitrary element to ± 1 essentially reduces to solving $a^2 - 13b^2 = \pm 4$ in $\mathbb{Z}$. This gives us the fundamental unit $\epsilon = \frac{3}{2} + \frac{\sqrt{13}}{2}$, we can now take ϵ and raise it to successive powers to find the first unit that lies completely in $\mathbb{Z}[\sqrt{13}]$, this yields $\epsilon^6 = 649 + 180\sqrt{13}$, giving us our solution. However, using the fundamental unit route to solve this problem seems like an artificial and long winded excuse to use fundamental units for what is really a basic Pell equation all of whose solutions (from elementary number theory) are given by the continued fraction method.

$\square$

Solution 3.

No, it is not neccessarily true that such a set of algebraic integers is finite. As a counterexample consider the family $\{(\sqrt{2}-1)^n\}_{n \in \mathbb{N}}$. For any $n \in \mathbb{N}$, $(\sqrt{2}-1)^n \in \mathbb{Z}[\sqrt{2}]$, hence $(\sqrt{2}-1)^n$ is an algebraic integer of degree at most 2. Furthermore as $0 < \sqrt{2}-1 < 1$, we have that $|(\sqrt{2}-1)^n| < 1$. Also, if $(\sqrt{2}-1)^i = (\sqrt{2}-1)^j$ we then have $(\sqrt{2}-1)^{(i-j)} = 1$ which implies $i = j$. Thus we truly do have an infinite family. $\square$