Solution 1.
Let $K \subseteq L \subseteq M$ be finite separable extensions of fields. Let $\sigma_1, \dots, \sigma_n$ be the distinct K-embeddings of L into Ω, and let $\tau_1, \dots, \tau_m$ be the distinct L-embeddings of M into Ω, where Ω is the galois closure of M over K. We then have that Ω/K is galois, and every map σ_i, τ_j extends to an automorphism of Ω allowing us to compose maps. Now (by corollary 2.19 in Milne)

$$Tr_{L/K} \circ Tr_{M/L} = \sum_{i=1}^{n} \sigma_i \circ (\sum_{j=1}^{m} \tau_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} \sigma_i \circ \tau_j$$

The last equality follows as σ_is are homomorphisms. Now each $\sigma_i \circ \tau_j$ is a K-embedding of M into Ω and as our extensions are separable, the number of such mappings is $mn = [M : L][L : K] = [M : K]$. To prove that $Tr_{M/K} = Tr_{L/K} \circ Tr_{M/L}$ it thus suffices to show that the $\sigma_i \circ \tau_j$ are all distinct when restricted to M. Note that if $\sigma_i \circ \tau_j = \sigma_x \circ \tau_y$ on M then $\sigma_i = \sigma_x$ on L as τ_j and τ_y are the identity on L. But as all the σ_is were distinct we have that $i = x$ which further implies that $\tau_j = \tau_y$ on M, but then as the τ_js were distinct we have that $j = y$. Thus, the $\sigma_i \circ \tau_j$ are all distinct as required. □

Solution 2.
Let $L = \mathbb{F}_2(x)$ and $K = \mathbb{F}_2(t)$ (so K is the field of rational functions in t with coefficients from $\mathbb{F}_2$), where $x^2 - t = 0$ (note that the polynomial $X^2 - t$ over K has a single root with multiplicity 2). Now, L/K is a non-separable extension. Using $\{1, x\}$ as a basis for L over K we then have, by definition

$$Disc(L/K) = \begin{vmatrix} Tr_{L/K}(1.1) & Tr_{L/K}(1.x) \\ Tr_{L/K}(x.1) & Tr_{L/K}(x.x) \end{vmatrix}$$

which by definition of the trace and using the fact that $x^2 = t$, is

$$= \begin{vmatrix} (2) & (0) \\ (0) & (2t) \end{vmatrix}$$

$$= 4t^2$$

but we are in $\mathbb{F}_2(t)$, so

$$= 0$$

□

Solution 3.
We claim that the ring of integers of $\mathbb{Q}(\alpha)$, where α is a root of the polynomial x^3+x^2-1, is $\mathbb{Z}[\alpha]$. To show this it suffices to show that $D(1, \alpha, \alpha^2)$ is squarefree (cf. remark 2.24 in Milne). We claim that $D(1, \alpha, \alpha^2) = -23$.

Preliminarily note that if we let $\alpha = \alpha_1$ and let α_2, α_3 be the galois conjugates of α_1 over the galois closure of $\mathbb{Q}(\alpha)$ then we have that

$$\alpha_1 + \alpha_2 + \alpha_3 = -1 \quad (1)$$
$$\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 = 0 \quad (2)$$
$$\alpha_1\alpha_2\alpha_3 = 1 \quad (3)$$

Furthemore combining (1), (2) and (3) we also obtain that

$$\begin{aligned}
\alpha_1^2+\alpha_2^2+\alpha_3^2 &= 1 && (4)\\
\alpha_1^2\alpha_2+\alpha_1^2\alpha_3+\alpha_2^2\alpha_1+\alpha_2^2\alpha_3+\alpha_3^2\alpha_1+\alpha_3^2\alpha_2 &= -3 && (5)\\
\alpha_1^2\alpha_2^2+\alpha_1^2\alpha_3^2+\alpha_2^2\alpha_3^2 &= 2 && (6)\\
\alpha_1\alpha_2\alpha_3^2+\alpha_1\alpha_2^2\alpha_3+\alpha_1^2\alpha_2\alpha_3 &= -1 && (7)\\
\alpha_1^2\alpha_2^2\alpha_3+\alpha_1^2\alpha_2\alpha_3^2+\alpha_1\alpha_2^2\alpha_3^2 &= 0 && (8)\\
\alpha_1^3 &= 1-\alpha_1^2 && (9)\\
\alpha_2^3 &= 1-\alpha_2^2 && (10)\\
\alpha_3^3 &= 1-\alpha_3^2 && (11)
\end{aligned}$$

Now (by proposition 2.23 in Milne) we have that

$$\begin{aligned}
D(1,\alpha,\alpha^2) &= \prod_{1\le i<j}(\alpha_i-\alpha_j)^2\\
&= (\alpha_3-\alpha_2)^2(\alpha_3-\alpha_1)^2(\alpha_2-\alpha_1)^2\\
&= (\alpha_2^2+\alpha_3^2-2\alpha_2\alpha_3)(\alpha_3^2+\alpha_1^2-2\alpha_1\alpha_3)(\alpha_2^2+\alpha_1^2-2\alpha_2\alpha_1)\\
&\quad \text{using (3) and (4) we then have}\\
&= (1-\alpha_1^2-\frac{2}{\alpha_1})(1-\alpha_2^2-\frac{2}{\alpha_2})(1-\alpha_3^2-\frac{2}{\alpha_3})\\
&= \frac{1}{\alpha_1\alpha_2\alpha_3}(\alpha_1-\alpha_1^3-2)(\alpha_2-\alpha_2^3-2)(\alpha_3-\alpha_3^3-2)\\
&\quad \text{using (3), (9), (10) and (11) we then have}\\
&= (\alpha_1^2+\alpha_1-3)(\alpha_2^2+\alpha_2-3)(\alpha_3^2+\alpha_3-3)\\
&= -27+9(\alpha_1+\alpha_2+\alpha_3)+9(\alpha_1^2+\alpha_2^2+\alpha_3^2)\\
&\quad -3(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3)-3(\alpha_1^2\alpha_2+\alpha_1^2\alpha_3+\alpha_2^2\alpha_1+\alpha_2^2\alpha_3+\alpha_3^2\alpha_1+\alpha_3^2\alpha_2)\\
&\quad -3(\alpha_1^2\alpha_2^2+\alpha_1^2\alpha_3^2+\alpha_2^2\alpha_3^2)+(\alpha_1\alpha_2\alpha_3^2+\alpha_1\alpha_2^2\alpha_3+\alpha_1^2\alpha_2\alpha_3)\\
&\quad +(\alpha_1^2\alpha_2^2\alpha_3+\alpha_1^2\alpha_2\alpha_3^2+\alpha_1\alpha_2^2\alpha_3^2)+(\alpha_1\alpha_2\alpha_3)+(\alpha_1\alpha_2\alpha_3)^2\\
&\quad \text{now using (1) through (9) we then have}\\
&= -27+9(-1)+9(1)-3(0)-3(-3)-3(2)+(-1)+(0)+1+(1)^2\\
&= -23
\end{aligned}$$

as required. □

Solution 4.
Let α be as stated in the problem, we then have that

$$\alpha^3=\alpha+4 \tag{1}$$

Note that $\gamma=\frac{\alpha(\alpha+1)}{2}\notin\mathbb{Z}[\alpha]$ (this follows from the fact that in the vector space $\mathbb{Q}(\alpha)$ we can write γ as a unique linear combination of $1,\alpha,\alpha^2$ over $\mathbb{Q}$). We claim that γ is integral over $\mathbb{Z}$, with minimum polynomial X^3-X^2-3X-2. The following calculation verifies this assertion

$$\begin{aligned}
\gamma^3-\gamma^2-3\gamma-2 &= \frac{\alpha^3(\alpha+1)^3}{8}-\frac{\alpha^2(\alpha+1)^2}{4}-\frac{3\alpha(\alpha+1)}{2}-2\\
&= \frac{(\alpha+4)(\alpha^3+3\alpha^2+3\alpha+1)}{8}-\frac{\alpha(\alpha^3)+2\alpha^3+\alpha^2}{4}-\frac{3\alpha^2-3\alpha}{2}-2\\
&\text{using (1) we then have}\\
&= \frac{(\alpha+4)(4\alpha+3\alpha^2+5)}{8}-\frac{\alpha(\alpha+4)+2(\alpha+4)+\alpha^2}{4}-\frac{3\alpha^2+3\alpha}{2}-2\\
&\text{using (1) again we have}\\
&= \frac{16\alpha^2+24\alpha+32}{8}-\frac{2\alpha^2+6\alpha+8}{4}-\frac{3\alpha^2+3\alpha}{2}-2\\
&= 2\alpha^2+3\alpha+2+\frac{-2\alpha^2-6\alpha-8-6\alpha^2-6\alpha}{4}\\
&= 2\alpha^2+3\alpha+2+\frac{-8\alpha^2-12\alpha-8}{4}\\
&= 0
\end{aligned}$$

as required. Thus, γ is integral over $\mathbb{Z}$ but $\gamma \notin \mathbb{Z}[\alpha]$ which implies that $\mathbb{Z}[\alpha]$ is not the ring of integers.

We claim that the ring of integers is $\mathbb{Z}[\gamma]$ i.e $\{1,\gamma,\gamma^2\}$ is an integral basis. Clearly $\mathbb{Z}[\gamma]$ is contained in the ring of integers, thus, it suffices to show that $D(1,\gamma,\gamma^2)$ is squarefree (cf. remark 2.24 in Milne). Now (by proposition 2.33 in Milne)

$$D(1,\gamma,\gamma^2) = disc(X^3-X^2-3X-2)$$

Using the fact that $disc(X^3+aX^2+bX+c) = -27c^2+18cab+a^2b^2-4a^3c-4b^3$ (cf. end of example 2.34 in Milne), we get that

$$D(1,\gamma,\gamma^2) = -107$$

which is prime and thus squarefree as required. □

Solution 5.
Let α be as stated in the problem. We claim that the ring of integers is $\mathbb{Z}[\alpha]$, i.e $\{1,\alpha,\alpha^2\}$ forms an integral basis. Using Maple we obtain that

$$D(1,\alpha,\alpha^2) = 2^2.223$$

Note that 223 is a rational prime. If we let $\mathcal{O}$ denote the ring of integers of $\mathbb{Q}(\alpha)$, then (by remark 2.24 in Milne)

$$D(1,\alpha,\alpha^2) = 2^2.223 = (\mathcal{O}:\mathbb{Z}[\alpha])^2.disc(\mathcal{O}/\mathbb{Z})$$

Thus, $\mathcal{O}:\mathbb{Z}[\alpha] \in \{1,2\}$. However, by Stickelberger's theorem $disc(\mathcal{O}/\mathbb{Z}) \equiv 0$ or 1 mod 4, as $223 \equiv -1 \mod 4$ this forces $\mathcal{O}:\mathbb{Z}[\alpha]=1$, which in turn implies that $\mathbb{Z}[\alpha]$ is the ring of integers.

By the remarks above it thus also follows that the prime factorization of the discriminant of this ring of integers is $2^2.223$. □