Solution 1.

(a) $K = \mathbb{Q}(\sqrt{-163})$, as $-163 \equiv 1 \mod 4$, $O_K = \mathbb{Z}[\frac{1+\sqrt{-163}}{2}]$ and $|\Delta_K| = |-163| = 163$. So the Minkowski bound is $B_K \leq \frac{2!}{2^2}\frac{4}{\pi}\sqrt{163} < 9$. Thus our representative ideals may have norms $1, 2, 3, 4, 5, 6, 7, 8$. So we must factorize the primes $2, 3, 5, 7$. Now the minimal polynomial for $\frac{1+\sqrt{-163}}{2}$ is $x^2 - x + 41$, which is irreducible mod 2,3,5 and 7. Thus, $(2), (3), (5), (7)$ are prime ideals in O_K and the class number of O_K is 1

(b) $K = \mathbb{Q}(\sqrt{-10})$, as $-10 \not\equiv 1 \mod 4$, $O_K = \mathbb{Z}[\sqrt{-10}]$ and $|\Delta_K| = |-(4).(10)| = 40$. So the Minkowski bound is $B_K \leq \frac{2!}{2^2}\frac{4}{\pi}\sqrt{40} < 5$. Thus our representative ideals may have norms $1, 2, 3, 4$. So we must factorize the primes $2, 3$. Now the minimal polynomial of $\sqrt{-10}$ is $x^2 + 10$ and $x^2 + 10 = x.x \mod 2$ and $x^2 + 10$ is irreducible mod 3. Thus $(2) = (2, \sqrt{-10})^2$ and (3) is prime. Note that $(2, \sqrt{-10})$ cannot be principal as that would imply the existence of a non-unit, non-zero element with norm that divided $N((2, \sqrt{-10})) = 2$ but this is clearly not possible as for a non-unit, non-zero element $a + b\sqrt{-10}$ in O_K, $N(a + b\sqrt{-10}) = a^2 + 10b^2 > 2$. Thus, the class number of O_K must be 2.

(c) $K = \mathbb{Q}(\sqrt{14})$ as $14 \not\equiv 1 \mod 4$, $O_K = \mathbb{Z}[\sqrt{14}]$ and $|\Delta_K| = |(4).(14)| = 56$. So the Minkowski bound is $B_K \leq \frac{2!}{2^2}\sqrt{56} < 4$. Thus, our representative ideals may have norms $1, 2, 3$. So we must factorize the primes $2, 3$. Now the minimal polynomial of $\sqrt{14}$ is $x^2 - 14$ and $x^2 - 14 = x.x \mod 2$ and $x^2 - 14$ is irreducible mod 3. So $(2) = (2, \sqrt{14})^2 = (4 + \sqrt{14})^2$ (as $(4 + \sqrt{14})(4 - \sqrt{14}) = 2$ so $(2, \sqrt{14}) \subseteq (4 + \sqrt{14})$ and clearly $(4 + \sqrt{14}) \subseteq (2, \sqrt{14})$) and (3) is prime. Thus, the class number of O_K must be 1.

□

Solution 2.

Let $K = \mathbb{Q}\sqrt{-23}$, as $-23 \equiv 1 \mod 4$, so $O_K = \mathbb{Z}[\gamma]$, where $\gamma = \frac{1+\sqrt{-23}}{2}$; and $|\Delta_K| = |-23| = 23$. So the Minkowski bound is $B_K \leq \frac{2!}{2^2}\frac{4}{\pi}\sqrt{23} < 4$. Thus, our representative ideals may have norms $1, 2, 3$. So we must factorize the primes $2, 3$. Now the minimal polynomial of γ is $x^2 - x + 6$ and $x^2 - x + 6 = x(x + 1) \mod 2$ and $x^2 - x + 6 = x(x - 1) \mod 3$. So, if we set $\mathfrak{p} = (2, \gamma), \mathfrak{q} = (2, \gamma + 1), \mathfrak{g} = (3, \gamma), \mathfrak{h} = (3, \gamma - 1)$ then $(2) = \mathfrak{pq}$ and $(3) = \mathfrak{gh}$. Note that $\mathfrak{p}$ can't be principal as that would imply the existence of a non-zero, non-unit element in O_K that has norm $N((2, \gamma)) = 2$, but if $a + b\gamma$ is a non-zero, non-unit in O_K then $N(a + b\gamma) = (a + \frac{b}{2})^2 + 23(\frac{b}{2})^2 > 2$. Further observe that $\gamma^2 = \gamma - 6$, so we get that

$$\mathfrak{p}^2 = (4, \gamma^2, 2\gamma) = (4, \gamma - 6, 2\gamma) = (4, \gamma + 2, 2\gamma) = (4, \gamma + 2)$$

and that

$$\begin{aligned}
\mathfrak{p}^3 &= (4, \gamma + 2)(2, \gamma) = (8, 4\gamma, 2\gamma + 4, \gamma^2 + 2\gamma) = (8, 2\gamma + 4, \gamma^2 + 2\gamma) = (8, 2\gamma + 4, 3\gamma - 6) \\
&= (8, 2\gamma - 4, 3\gamma - 6) = (8, 2(\gamma - 2), 3(\gamma - 2), (3\gamma - 6) - (2\gamma - 4)) = (8, \gamma - 2) \\
&= (8, \gamma - 2, (\gamma - 2)(\gamma - 3)) = (8, \gamma - 2, 4\gamma) = (\gamma - 2)
\end{aligned}$$

Furthermore, $\mathfrak{pg} = (6, 2\gamma, 3\gamma, \gamma^2) = (6, 2\gamma, 3\gamma, \gamma - 6) = (6, \gamma) = (\gamma(1-\gamma), \gamma) = (\gamma)$ and consequently we have that in the classgroup $[\mathfrak{p}]^3 = [1]$, $[\mathfrak{q}] = [\mathfrak{g}] = [\mathfrak{p}]^2$ and $[\mathfrak{h}] = [\mathfrak{p}]$. Thus the class group is generated by the representative of $\mathfrak{p}$ and has order 3. □

Solution 3.

Let K be a number field, O_K its ring of integers. Let h denote the class number of O_K. We know that h is finite, so we may pick a finite set of ideal representatives $\mathfrak{a}_1, \ldots, \mathfrak{a}_h$ for the class group, where each $\mathfrak{a}_i$ is an integral ideal in O_K (cf. Theorem 4.3 in Milne). It suffices to find $L|K$ such that each $\mathfrak{a}_i O_L$ is principal as for any other ideal $\mathfrak{b} \subseteq O_K$ we have that $\mathfrak{b} = \mathfrak{a}_i(y_i)$ for some $y_i \in K$ and consequently if $\mathfrak{a}O_L$ is principal then so must $\mathfrak{b}O_L$.

Note that as h is the order of the group, we have that for each i, $\mathfrak{a}_i^h = (x_i)$ for some $x_i \in O_K$ (note that $x_i \in O_K$ as we picked $\mathfrak{a}_i$ to be integral). Now as $x_i \in O_K$ we have that x_i is integral over $\mathbb{Z}$, but $x_i^{\frac{1}{h}}$ is integral over O_K so by 'transitivity of integrality' $x_i^{\frac{1}{h}}$ is integral over $\mathbb{Z}$. We claim that $L = K(x_1^{\frac{1}{h}}, \ldots, x_h^{\frac{1}{h}})$ is the required extension. Let O_L denote the ring of integers of L. Note that from our remarks above each $x_i^{\frac{1}{h}} \in O_L$, furthermore we have that $\mathfrak{a}_i^h = (x_i)$ which implies that working in O_L we have $\mathfrak{a}_i^h O_L = (x_i^{\frac{1}{h}})^h O_L$, now using unique factorization of fractional ideals of O_L it easily follows that $(\mathfrak{a}O_L = (x_i^{\frac{1}{h}})O_L$, as required. (To see the last equality take any fractional prime ideal dividing $\mathfrak{a}_i$ then this prime ideal must divide $(x_i^{\frac{1}{h}})^h$ so it must divide $(x_i^{\frac{1}{h}})$. Conversely any prime ideal that divides $(x_i^{\frac{1}{h}})$ must by the same argument also divide $\mathfrak{a}_i$.) □

Solution 4.

(a) Let $K = \mathbb{Q}(i, \sqrt{5})$ and O_K be the ring of integers of K. Let $L = \mathbb{Q}(\sqrt{-5})$ and $M = \mathbb{Q}(i)$ with their ring of integers being $O_L = \mathbb{Z}[\sqrt{-5}]$ and $O_M = \mathbb{Z}[i]$ respectively. From HW 7 we know that the only prime ideals from $\mathbb{Z}$ that ramify in O_K are (2) and (5). From this it follows that the only prime ideals in O_L that may possibly ramify in O_K are those that occur in the factorization of (2) and (5) in O_L (because if a prime ideal from O_L ramified in O_K then all the primes in $\mathbb{Z}$ that it lies over must also ramify). Factorizing in O_L we have that, $(2) = (2, 1 + \sqrt{-5})^2$ and $(5) = (5, \sqrt{-5})^2$ (both factorizations were obtained by looking at $x^2 + 5$ modulo 2 and 5). To prove the required assertion it thus suffices to show that in the factorization of (2) and (5) in O_K no ideal factor occurs to a fourth or higher power (as then $(2, 1 + \sqrt{-5})$ and $(5, \sqrt{-5})$ cannot possibly ramify in O_K). Observe that in O_M, $(2) = (2, 1 + i)^2$ and that $(5) = (5, i + 2)(5, i - 2)$ (these factorizations were obtained by looking at $x^2 + 1$ modulo 2 and 5).

Note that $(5, i + 2) \neq (5, i - 2)$ as otherwise $i - 2 \in (5, i + 2)$ which implies that $(5, i+2) = (5, i-2) = O_M$, which is absurd. Observe that K is a degree 2 extension over M, thus by Theorem 3.36 in Milne, working in O_K, $(5, i + 2)$ may factor into at most two prime factors (not neccessarily distinct, in fact it is easy to see that

they wont be distinct). Similarly $(5, i-2)$ may also have at most two prime factors in O_K. The sets of prime factors of $(5, i+2)$ and $(5, i-2)$ must be disjoint (as otherwise the common factor would contain $5, i+2, i-2$ and following the same line of reasoning as earlier, would consequently have to be all of O_K). Thus, we have just shown that in the factorization of (5) in O_K no factor shows up to a fourth or higher power.

To obtain the required result for (2), observe that it suffices to show that $(2, 1+i)$ doesn't ramify in O_K. Note that $\gamma = \frac{1+\sqrt{5}}{2}$ has minimal polynomial $x^2 - x - 1$ and thus $\gamma \in O_K$. Observe that (by proposition 2.25 in Milne) $Disc(O_M(\gamma)/O_M)$ is the ideal generated by $\begin{vmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ 1 & \frac{1-\sqrt{5}}{2} \end{vmatrix}^2 = 5$. Consequently by lemma 2.22 in Milne, $Disc(O_K/O_M)$ divides (5) as ideals in O_M (in fact it is easy to see that $Disc(O_K/O_M) = (5)$ and that consequently $O_K = \mathbb{Z}[i, \gamma]$). Clearly $(2, 1+i)$ doesn't divide (5) in O_M (as otherwise $(2, 1+i) = O_M$). Thus, (by Theorem 3.37 in Milne) $(2, 1+i)$ doesn't ramify in O_K, as required.

(b) Note that the class number of $L = \mathbb{Q}(\sqrt{-5})$ is 2 thus we know that the Hilbert class field of L is a degree 2 extension over L. However, $K = \mathbb{Q}(i, \sqrt{5})$ is a degree 2 and clearly abelian extension of L, thus by the uniqueness of the Hilbert class field, K must be the Hilbert class field of L. Now any other abelian unramified extensions of L must be strictly contained in K and thus have degree less than 2 over L. Clearly, there is no such non trivial extension.

□