Solution 1.

Basis vectors over $\mathbb{R}^3$: $\{(5,0,0),(0,\alpha+3,0),(0,0,\alpha^2+1)\}$.

Note that the conjugates of α are $\alpha\omega$ and $\alpha\omega^2$ where ω is a primitive third root of unity. Further observe that $\mathfrak{a}$ lies over (5) so its norm must be either 5 or 25. But $\mathfrak{a}$ also lies over $(\alpha+3)$ which has norm $(\alpha+3)(\alpha\omega+3)(\alpha\omega^2+3)=30$, thus $N(\mathfrak{a})=5$.

Using Maple we have that $Disc(O_K)=-243$. The negative discriminant shows that $r=s=1$ (since the sign of the discriminant is $(-1)^s$). Now

$$Vol(\sigma(\mathfrak{a}))=(2i)^{-s}\begin{vmatrix} 5 & \alpha+3 & \alpha^2+1 \\ 5 & \alpha\omega+3 & \alpha^2\omega+1 \\ 5 & \alpha\omega^2+3 & \alpha^2\omega^2+1 \end{vmatrix}=\frac{1}{2i}45\sqrt{3}i=\frac{45\sqrt{3}}{2}$$

And our theorem tells us that $Vol(\sigma(\mathfrak{a}))=2^{-s}\mathbb{N}\mathfrak{a}|\Delta_K|^{\frac{1}{2}}=\frac{1}{2}5|-243|^{\frac{1}{2}}=\frac{45\sqrt{3}}{2}$, hence verified.

□

Solution 2.

(a) $r=3$

$$S(7)=\{\vec{x}\in\mathbb{R}^3 : \|\vec{x}\|=|x_1|+|x_2|+|x_3|\le 7\}$$

This is the volume of the space enclosed by a 'double pyramid'. Consider just one half of this space which is a pyramid of square base of area $(7\sqrt{2})^2$ and height 7. Thus the total volume of $S(7)$ is $2\left(\frac{1}{3}.7.(7\sqrt{2})^2\right)=\frac{2^2.7^3}{3}$. Our theorem tells us that the volume should be $2^r4^{-s}(2\pi)^s\frac{t^n}{n!}=2^3.1.1.\frac{7^3}{3!}=\frac{2^2.7^3}{3}$, hence verified.

(b) $r=s=1$

$$S(7)=\{\vec{x}\in\mathbb{R}\times\mathbb{C} : \|\vec{x}\|=|x_1|+2\sqrt{x_2^2+x_3^2}\le 7$$

The volume enclosed by this space is equivalent to evaluating the integral

$$2\iint_D 7-2\sqrt{x_2^2+x_3^2}\,dA$$

where D is the region $x_2^2+x_3^2\le\frac{49}{4}$. Switching to polar coordinates we obtain

$$2\int_0^{2\pi}\int_0^{\frac{7}{2}}(7-2r)r\,dr\,d\theta=\frac{7^3\pi}{6}$$

Our theorem tells us that the volume should be $2^r4^{-s}(2\pi)^s\frac{t^n}{n!}=\frac{2}{4}.(2\pi)\frac{7^3}{3!}=\frac{7^3\pi}{6}$, hence verified.

Now note that $\frac{7^3\pi}{6} \approx 179.6$ and $2^3\frac{45\sqrt{3}}{2} \approx 311.8$. So $Vol(S(7)) \ngeq 2^3 Vol(\sigma(\mathfrak{a}))$ and hence $S(7)$ is <u>not</u> large enough to ensure that it contains a point of the lattice $\sigma(\mathfrak{a})$.

$\square$