Solution 1.

Throughout this problem, K will always denote an imaginary quadratic number field. So in O_K the group of units is equal to the finite subgroup of torsion elements, which we know is cyclic and hence generated by a primitive root of unity. So to find $\mu(K)$ we essentially need to find which ζ_m is in K. From class/the statement of the problem we know that $m = 1, 2, 3, 4$. Note that $\pm 1 \in \mathbb{Q}$ so we really only need to consider the cases $m = 3, 4, 6$. Note that $\mathbb{Q}(\zeta_6) = \mathbb{Q}(\zeta_3)$. If $\zeta_3 \in K$ then $K = \mathbb{Q}(\zeta_3)$ (as $|\mathbb{Q}(\zeta_3) : \mathbb{Q}| = 2 = |K : \mathbb{Q}|$ and $\mathbb{Q}(\zeta_3) \subseteq K$). Thus we have just shown that $\zeta_3 \in K$ iff $K = \mathbb{Q}(\zeta_3)$, furthermore note that $\mathbb{Q}(\zeta_3) = \mathbb{Q}(\sqrt{-3})$, thus $\zeta_3 \in K$ iff $K = \mathbb{Q}(\sqrt{-3})$. On the other hand if $\zeta_4 \in K$ then $i \in K$ and consequently (using the same reasoning as we did for ζ_3) $\zeta_4 \in K$ iff $K = \mathbb{Q}(i)$.

In conclusion for imaginary quadratic fields the only units/roots of unity in $\mu(K)$ are ± 1 except in the two cases $K = \mathbb{Q}(i)$ where they are $\pm 1, \pm i$ and $K = \mathbb{Q}(\sqrt{-3})$ where they are $\pm 1, \pm\zeta_3, \pm\zeta_3^2$.

Alternate 'brute force' proof: Let $K = \mathbb{Q}(\sqrt{-d})$, $d > 0$ and squarefree.

Case 1: $-d \equiv 1 \mod 4$

So $O_K = \{a + b(1+\sqrt{-d})/2 \,:\, a, b \in \mathbb{Z}\}$. Taking norms and noting that an element in O_K is a unit iff its norm is ± 1 we see that the units in O_K are given by solutions to the equation

$$(2a + b)^2 + db^2 = 4$$

But if $d > 4$ then we have that b must be 0 and consequently $a = \pm 1$. If $d = 3$ we have that $b \in \{0, \pm 1\}$. Enumerating each case we see that the only units in O_K ($K = \mathbb{Q}(\sqrt{-3})$) are $\{\pm 1, \pm\zeta_3, \pm\zeta_3^2\}$.

Case 2: $-d \not\equiv 1 \mod 4$

In this case $O_K = \{a + b\sqrt{-d} \,:\, a, b \in \mathbb{Z}\}$. Taking norms and noting that an element in O_K is a unit iff its norm is ± 1 we see that the units in O_K are given by solutions to the equation

$$a^2 + db^2 = 1$$

But if $d > 1$ then we have that b must be 0 and consequently $a = \pm 1$. If $d = 1$ we have that $b \in \{0, \pm 1\}$. Enumerating each case we see that the only units in O_K ($K = \mathbb{Q}(i)$) are $\{\pm 1, \pm i\}$.

In conclusion for imaginary quadratic fields the only units are ± 1 except in the two cases $K = \mathbb{Q}(i)$ where the units are $\pm 1, \pm i$ and $K = \mathbb{Q}(\sqrt{-3})$ where the units are $\pm 1, \pm\zeta_3, \pm\zeta_3^2$.

$\square$

Solution 2.

$\mathbb{Q}(\sqrt{11})$

$11 \not\equiv 1 \mod 4$, so we compute the continued fraction of $\sqrt{11}$.

$$\begin{aligned} a_0 &= \left[\sqrt{11}\right] = \left[3 + (\sqrt{11} - 3)\right] = 3 \\ a_1 &= \left[\frac{1}{\sqrt{11} - 3}\right] = \left[\frac{\sqrt{11} + 3}{2}\right] = \left[3 + \frac{\sqrt{11} - 3}{2}\right] = 3 \\ a_2 &= \left[\frac{2}{\sqrt{11} - 3}\right] = \left[\sqrt{11} + 3\right] = \left[6 + (\sqrt{11} - 3)\right] = 6 \\ a_3 &= a_1 \\ a_4 &= a_2 \end{aligned}$$

Thus, the period length is 2, so the fundamental unit is given by the second convergent. The first two convergents are $3/1, 10/3$. So $\epsilon = 10 - 3\sqrt{11}$. Furthermore we also compute $p_n^2 - 11q_n^2$ at every step and note that $10^2 - 11(3)^2 = 1$ is the first instance when we get ± 1.

$\mathbb{Q}(\sqrt{19})$

$19 \not\equiv 1 \mod 4$, so we compute the continued fraction of $\sqrt{19}$.

$$\begin{aligned} a_0 &= \left[\sqrt{19}\right] = \left[4 + (\sqrt{19} - 4)\right] = 4 \\ a_1 &= \left[\frac{1}{\sqrt{19} - 4}\right] = \left[\frac{\sqrt{19} + 4}{3}\right] = \left[2 + \frac{\sqrt{19} - 2}{3}\right] = 2 \\ a_2 &= \left[\frac{3}{\sqrt{19} - 2}\right] = \left[\frac{\sqrt{19} + 2}{5}\right] = \left[1 + \frac{\sqrt{19} - 3}{5}\right] = 1 \\ a_3 &= \left[\frac{5}{\sqrt{19} - 3}\right] = \left[\frac{\sqrt{19} + 3}{2}\right] = \left[3 + \frac{\sqrt{19} - 3}{2}\right] = 3 \\ a_4 &= \left[\frac{2}{\sqrt{19} - 3}\right] = \left[\frac{\sqrt{19} + 3}{5}\right] = \left[1 + \frac{\sqrt{19} - 2}{5}\right] = 1 \\ a_5 &= \left[\frac{5}{\sqrt{19} - 2}\right] = \left[\frac{\sqrt{19} + 2}{3}\right] = \left[2 + \frac{\sqrt{19} - 4}{3}\right] = 2 \\ a_6 &= \left[\frac{3}{\sqrt{19} - 4}\right] = \left[\sqrt{19} + 4\right] = \left[8 + (\sqrt{19} - 4)\right] = 8 \\ a_7 &= a_1 \\ a_8 &= a_2 \\ a_9 &= a_3 \end{aligned}$$

Thus, the period length is 6, so the fundamental unit is given by the 6th convergent. convergents are: $4/1, 9/2, 13/3, 48/11, 61/14, 170/39$. Thus, $\epsilon = 170 - 39\sqrt{19}$. Furthermore we also compute $p_n^2 - 11q_n^2$ at every step and note that $170^2 - 19(39)^2 = 1$ is the first instance when we get ± 1.

$\mathbb{Q}(\sqrt{22})$

$22 \not\equiv 1 \mod 4$, so we compute the continued fraction of $\sqrt{22}$.

$$\begin{aligned}
a_0 &= \left\lfloor \sqrt{22} \right\rfloor = \left\lfloor 4 + (\sqrt{22} - 4) \right\rfloor = 4 \\
a_1 &= \left\lfloor \frac{1}{\sqrt{22} - 4} \right\rfloor = \left\lfloor \frac{\sqrt{22} + 4}{6} \right\rfloor = \left\lfloor 1 + \frac{\sqrt{22} - 2}{6} \right\rfloor = 1 \\
a_2 &= \left\lfloor \frac{6}{\sqrt{22} - 2} \right\rfloor = \left\lfloor \frac{\sqrt{22} + 2}{3} \right\rfloor = \left\lfloor 2 + \frac{\sqrt{22} - 4}{3} \right\rfloor = 2 \\
a_3 &= \left\lfloor \frac{3}{\sqrt{22} - 4} \right\rfloor = \left\lfloor \frac{\sqrt{22} + 4}{2} \right\rfloor = \left\lfloor 4 + \frac{\sqrt{22} - 4}{2} \right\rfloor = 4 \\
a_4 &= \left\lfloor \frac{2}{\sqrt{22} - 4} \right\rfloor = \left\lfloor \frac{\sqrt{22} + 4}{3} \right\rfloor = \left\lfloor 2 + \frac{\sqrt{22} - 2}{3} \right\rfloor = 2 \\
a_5 &= \left\lfloor \frac{3}{\sqrt{22} - 2} \right\rfloor = \left\lfloor \frac{\sqrt{22} + 2}{6} \right\rfloor = \left\lfloor 1 + \frac{\sqrt{22} - 4}{6} \right\rfloor = 1 \\
a_6 &= \left\lfloor \frac{6}{\sqrt{22} - 4} \right\rfloor = \left\lfloor \frac{\sqrt{22} + 4}{1} \right\rfloor = \left\lfloor 8 + (\sqrt{22} - 4) \right\rfloor = 8 \\
a_7 &= a_1
\end{aligned}$$

Thus, the period length is 6, so the fundamental unit is given by the 6th convergent, the first 6 convergents are $4/1, 5/1, 14/3, 61/13, 136/29, 197/42$. Thus, $\epsilon = 197 - 42\sqrt{22}$. Furthermore we also compute $p_n^2 - 22q_n^2$ at every step and note that $197^2 - 22(42)^2 = 1$ is the first instance when we get ± 1.

□

Solution 3.

So we have that

$$2c = \rho^3 + \rho^{-3} \tag{1}$$
$$\beta^2 - \frac{c}{2}\beta - \frac{1}{2} = 0 \tag{2}$$
$$\rho^6 - 4\beta^2 - 4\beta^4 < 0 \tag{3}$$
$$|\Delta'| \leq 16(c^2 - 2\beta c + \beta^2)(1 - \beta^2) \tag{4}$$

(2) then gives us that $2\beta c = 4\beta^2 - 2$, substituting this into (4) we have that

$$|\Delta'| \leq 16(c^2 - 3\beta^2 + 2)(1 - \beta^2)$$

Now substituting for c (by using (1)) we have

$$|\Delta'| \leq (4\rho^6 + 4\rho^{-6} - 8 - 48\beta^2 + 32)(1 - \beta^2)$$

Furthermore, (3) gives us that $\rho^6 < 4\beta^2 + 4\beta^4$, using this in the last equation we have that

$$|\Delta'| < (4\rho^{-6} - 32\beta^2 + 16\beta^4 + 24)(1 - \beta^2)$$

but as $-1 \leq \beta = \cos(\theta) \leq 1$, we have that $\beta^4 \leq \beta^2$, so we then have that

$$|\Delta'| < (4\rho^{-6} - 32\beta^2 + 16\beta^2 + 24)(1 - \beta^2) = (4\rho^{-6} - 16\beta^2 + 24)(1 - \beta^2)$$

But again as $\cos(\theta) = \beta$ we have that $0 \leq \beta^2 \leq 1$ hence,

$$|\Delta'| < (4\rho^{-6} - 16(0) + 24)(1 - 0)$$

But note that $\epsilon = \rho^{-2}$, so we have that $|\Delta'| < 4\epsilon^3 + 24$, furthermore $\mathbb{Z}[\epsilon] \subseteq O_K$ so $|\Delta'| = |\Delta_K|$.(square of some integer), so $|\Delta_K| < |\Delta'| < 4\epsilon^3 + 24$ or that $\epsilon^3 > \frac{|\Delta_K| - 24}{4}$, as required.

Bonus problem: Let K be a real quadratic field and let ϵ be a fundamental unit. Since $-\epsilon$, $-\epsilon^{-1}$ and ϵ^{-1} are also fundamental units, we may suppose $\epsilon > 1$. Clearly, $\epsilon \notin \mathbb{Q}$, and as the norm of ϵ must be ± 1, we thus have that the conjugate of ϵ is $\pm\frac{1}{\epsilon}$. We claim that $\epsilon^2 > \Delta_K - 3$.

Let Δ' be the discriminant of the minimum equation of ϵ.

Case 1: The conjugate of ϵ is $\frac{1}{\epsilon}$

$$\begin{aligned} \Delta' &= \begin{vmatrix} 1 & \epsilon \\ 1 & \frac{1}{\epsilon} \end{vmatrix}^2 \\ &= (\epsilon - \frac{1}{\epsilon})^2 \\ &= \epsilon^2 + \frac{1}{\epsilon^2} - 2 \\ &< \epsilon^2 + \frac{1}{\epsilon^2} + 2 \end{aligned}$$

Case 2: The conjugate of ϵ is $\frac{-1}{\epsilon}$

$$\begin{aligned} \Delta' &= \begin{vmatrix} 1 & \epsilon \\ 1 & \frac{-1}{\epsilon} \end{vmatrix}^2 \\ &= (\epsilon + \frac{1}{\epsilon})^2 \\ &= \epsilon^2 + \frac{1}{\epsilon^2} + 2 \end{aligned}$$

We have thus just shown that $\Delta' \leq \epsilon^2 + \frac{1}{\epsilon^2} + 2$. Since $\Delta' = \Delta_K$.(square of some integer) we have that

$$\begin{aligned} \Delta_K &< \Delta' \\ &\leq \epsilon^2 + \frac{1}{\epsilon^2} + 2 \\ &< \epsilon^2 + 3 \end{aligned}$$

and thus we have that $\epsilon^2 > \Delta_K - 3$, as required. □

Solution 4.

Note that as $\Delta_K = -108$ is negative, K is a cubic number field with exactly one real embedding and thus there exists a unique fundamental unit $\epsilon > 1$. Furthermore we know that $\epsilon > \left(\frac{|\Delta_K|-24}{4}\right)^{\frac{1}{3}} \approx 2.758$. Observe that $Nm(\sqrt[3]{2}-1) = (\sqrt[3]{2}-1)(\sqrt[3]{2}\omega - 1)(\sqrt[3]{2}\omega^2 - 1) = 1$, where ω is a primitive third root of unity. Thus, $\frac{1}{\sqrt[3]{2}-1}$ is also a unit and an element of O_K. Furthermore, $\frac{1}{\sqrt[3]{2}-1} \approx 3.847$ and as $\frac{1}{\sqrt[3]{2}-1}$ is a power of ϵ we must have that $\epsilon = \frac{1}{\sqrt[3]{2}-1}$. □