12 Jan 2009 Algebraic Lie Theory at the Newton Institute.

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Derived categories and perverse sheaves I

Plan:
1. Introduce derived categories
2. t-structures and perverse sheaves
3. D-modules and Riemann-Hilbert correspondence
4. Weights, purity and the decomposition theorem
5. Applications. Perverse coherent sheaves.

$\mathcal{A}$ - abelian category
eg. R-mod for an algebra R.
sheaves of vector spaces on a topological space.

Problem: Many useful functors fail to preserve short exact sequences.

Keeping track of this failure has useful information.

classical approach: derived functors.

$\mathrm{Ext}^n(A,B)$: to compute:
1) Replace B by injective resolution
$I^\bullet: \; 0 \to I^0 \to I^1 \to \cdots$

2) compute $\mathrm{Hom}(A, I^\bullet)$, name it
$0 \to M^0 \to M^1 \to$

3) Take $H^n(M^\bullet) = \mathrm{Ext}^n(A,B)$.

<u>Fancier approach</u>:

Replace the category $\mathcal{A}$ by a new category in which $B \simeq I^\bullet$ (and also all other injective resolutions).
This new category is the derived category $D(\mathcal{A})$.

• ~~All~~ objects of $D(\mathcal{A})$ are chain complexes of objects of $\mathcal{A}$.
convention: complexes go up (cohomological notation).

objects of $\mathcal{A}$ regarded as chain complex concentrated in degree 1.

<u>Def</u> A map of chain complexes $f: C^\bullet \to D^\bullet$ is a <u>quasi-isomorphism</u> if all
$H^n(f): H^n(C) \xrightarrow{\sim} H^n(D)$ are
~~quasi~~ isomorphisms.

<u>morphisms in $D(\mathcal{A})$</u>:
Take all maps of chain complexes and formally invert all quasi-isomorphisms.

Note:

$$B \simeq I^\bullet \quad \text{in } D(\mathcal{A})$$

object of $\mathcal{A}$ ~~re~~ injective resolution of B

<u>Question</u> What's $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(C^\bullet, D^\bullet)$

usually work w/
$\mathcal{D}^b(A)$ = bounded derived category
= objects $C^\bullet$ in $\mathcal{D}(\mathcal{A})$ with $H^n(C^\bullet) = 0$ for $n \gg 0$ or $n \ll 0$.

<u>Back to question</u>: <u>Guess</u>:
1) Form the chain complex
$$\underline{\mathrm{Hom}}(C^\bullet, D^\bullet)^n = \bigoplus_{j-i=n} \mathrm{Hom}(C^i, D^j)$$
$\boxed{\text{WHAT ARE DIFFERENTIALS ?}}$
2) Take 0th cohomology $H^0(\underline{\mathrm{Hom}}(C^\bullet, D^\bullet))$

<u>Prop</u>: Assume either $D^\bullet$ is a complex of injective objects or $C^\bullet$ is a complex of projective objects. Then
$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})} \simeq \cancel{\mathrm{Hom}}\ H^0(\underline{\mathrm{Hom}}(C^\bullet, D^\bullet))$$
$$\simeq \left\{\begin{array}{l}\text{homotopy classes of chain}\\ \text{complex maps } C^\bullet \to D^\bullet\end{array}\right.$$

So $\mathrm{Ext}^n(A,B) = H^n(\underline{\mathrm{Hom}}(A, I^\bullet))$

$$= H^0(\underline{\mathrm{Hom}}(A, I[n]))$$

shift indexing of $I^\bullet$ by n.

So $\mathrm{Ext}^n(A,B) \simeq \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(A, B[n])$

Observations :

1) $D^b(\mathcal{A})$ is ~~are~~ usually not abelian (kernel, surjective etc. have no meaning)

2) Snake lemma: Say

$0 \to C^\bullet \to D^\bullet \to E^\bullet \to 0$ is a short exact sequence of chain complexes, then we have the long exact sequence of cohomology

$$\to H^n(C^\bullet) \to H^n(D^\bullet) \to H^n(E^\bullet) \xrightarrow{\partial} H^{n+1}(C^\bullet)$$

$$\to \cdots$$

∂ = boundary homomorphism / connecting map.

∂ doesn't come from a map of chain complexes but it does come from a map $E^\bullet \to C^\bullet[1]$ in $D^b(\mathcal{A})$.

The diagram

$C^\bullet \to D^\bullet \to E^\bullet \rightsquigarrow C^\bullet[1]$ is an instance of ~~the~~ a distinguished triangle

Morally: Distinguished triangles replace short exact sequences
- basic unit of thinking in $D^b(\mathcal{A})$.

<u>Properties</u> of <u>distinguished</u> <u>triangles</u>

<u>TR1</u> $A^\bullet \xrightarrow{id} A^\bullet \longrightarrow 0 \longrightarrow A[1]$ is a distinguished triangle

<u>TR2</u> Any $f: A^\bullet \to B^\bullet$ in $D^b(\mathcal{A})$ can be ~~completeti~~ completed to a distinguished triangle

$$A^\bullet \xrightarrow{f} B^\bullet \longrightarrow C^\bullet \longrightarrow A^\bullet[1]$$

(If f is a map of chain complexes then $C^\bullet$ (the "cone" of f) "contains" coker f and $(\ker f)[1]$)

<u>TR3</u> Rotation:

$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{h} A[1]$ is a distinguished triangle

$$\Longleftrightarrow$$

$B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{h} A[1] \xrightarrow{-f[1]} B^\bullet[1]$ is a distinguished triangle

<u>TR4</u> completing commutative diagrams with distinguished triangles.

<u>TR5</u> Octahedral axiom

<u>Thm</u> $D^b(\mathcal{A})$ satisfies TR1 – TR5

<u>Def</u> Any additive category w/ a specified collection of diagrams satisfying TR1 – TR5 is a <u>triangulated category</u>.

Fancier approach to derived functors:

$R\mathrm{Hom}(C^\bullet, D^\bullet)$

1) Replace $D^\bullet$ by a quasi-isomorphic complex of injectives (or $C^\bullet$ by quasi-isomorphic complex of projectives).

2) Form $\underline{\mathrm{Hom}}(C^\bullet, D^\bullet)$

3) Regard this chain complex as an object of D(abelian groups) or D(vector spaces)

Facts:

1) $R\mathrm{Hom}(C^\bullet, D^\bullet)$ well defined up to quasi-isomorphism

2) $R\mathrm{Hom}$ is a functor. It preserves distinguished triangles.

3) (snake lemma - generalization)
H^* (taking cohomology) takes any distinguished triangle to a long exact sequence in $\mathcal{A}$.

eg. apply $R\mathrm{Hom}(A^\bullet, \cdot)$ to
$$C^\bullet \longrightarrow D^\bullet \longrightarrow E^\bullet \longrightarrow C^\bullet[1]$$
to get a distinguished triangle

$$R\mathrm{Hom}(A^\bullet, C^\bullet) \longrightarrow R\mathrm{Hom}(A^\bullet, D^\bullet)$$

$$\hookrightarrow R\mathrm{Hom}(A^\bullet, E^\bullet) \longrightarrow$$

Apply H^* to get

$$\to \mathrm{Hom}(A, C^\bullet) \to \mathrm{Hom}(A^\bullet, D^\bullet) \to \mathrm{Hom}(A^\bullet, E^\bullet)$$

$(A, C^\bullet[1]) \longrightarrow$

Another example : Global sections

$\Gamma : Sh(X) \longrightarrow Vect$ } left exact

Sheaves of vector spaces on X

$R\Gamma : D^b(X) \longrightarrow D^b(Vect)$

Apply to constant sheaf $\underline{\mathbb{C}}_X$

$H^n(R\Gamma(\underline{\mathbb{C}}_X))$ - sheaf cohomology for nice spaces, singular or de Rham

$\simeq H^n(X, \mathbb{C})$

Poincaré - Verdier duality

$\mathbb{D} : D^b(X) \longrightarrow D^b(X)$, anti-equiv, $\mathbb{D}^2 \simeq id$

For an orientable manifold
$\mathbb{D} = RHom(-, \underline{\mathbb{C}}_X[\dim X])$

$H^n(R\Gamma(\mathbb{D}\underline{\mathbb{C}}_X))$

$R\Gamma \circ \mathbb{D} \simeq \mathbb{D} \circ R\Gamma$ compact manifold

$\mathbb{D}\underline{\mathbb{C}}_X \simeq \underline{\mathbb{C}}_X[\dim X]$

$H^{n+\dim X}(X, \mathbb{C}) \xrightarrow{\sim} H^n(X, \mathbb{C})^*$