13 Jan 2009 Algebraic Lie Theory at the Newton Institute

P. Achar Derived categories and perverse sheaves II

$\mathcal{A}$ - abelian category
$D^b(\mathcal{A})$ - bounded derived category

$\mathcal{A} \subset D^b(\mathcal{A})$: full abelian subcategory, i.e $Hom_{\mathcal{A}}(M,N) \simeq Hom_{D^b(\mathcal{A})}(M,N)$

Problem: Find other abelian subcategories inside $D^b(\mathcal{A})$

Def Let D be a triangulated category. Let $D^{\leq 0}$, $D^{\geq 0}$ be 2 full subcategories. Let $D^{\leq n} = D^{\leq 0}[-n]$, $D^{\geq n} = D^{\geq 0}[n]$. The pair $(D^{\leq 0}, D^{\geq 0})$ is a t-structure if

1) $Hom(D^{\leq 0}, D^{\geq 0}) = 0$
2) $D^{\leq 0} \subset D^{\leq 1}$, $D^{\geq 0} \supset D^{\geq 1}$
3) $\forall$ objects X there exists a distinguished triangle $A \to X \to B \rightsquigarrow$ w/ $A \in D^{\leq 0}$, $B \in D^{\geq 1}$

example 0 $D = D^b(\mathcal{A})$

$D^{\leq 0} = \{ C^\bullet \mid H^n(C^\bullet) = 0 \quad \forall n > 0 \}$

$D^{\geq 0} = \{ C^\bullet \mid H^n(C^\bullet) = 0 \quad \forall n < 0 \}$

<u>Thm</u> Let $\mathcal{D}$ be a triangulated category w/ t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. The full subcategory $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is an abelian category.

<u>Defn</u> $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is called the <u>heart</u> or <u>core</u> (French: coeur) of the t-structure.

<u>example</u> 0'

X = topological space w/ "topological stratification"

finite $\longrightarrow$ $X = \sqcup$ of smooth manifolds (<u>strata</u>) w/ conditions on tangent spaces

Typical example: X is a $\mathbb{C}$-variety
strata: orbits of some group action

<u>Def</u> A sheaf $\mathcal{F}$ on X is <u>constructible</u> if $\mathcal{F}|_S$ is a local system for all strata S.

$$\mathcal{D}^b_c(X) = \text{full subcategory of } \mathcal{D}^b(X) \text{ s.t } \left\{ \mathcal{F}^\bullet \,/\, H^n(\mathcal{F}^\bullet) \text{ is constructible for all } n \right\}$$

This is a full triangulated subcategory of $\mathcal{D}^b(X)$, i.e if $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ is a morphism in $\mathcal{D}^b_c(X)$ then the cone is also in $\mathcal{D}^b_c(X)$

$$D = D^b_c(X)$$

$$D^{\leq 0} = \{ \mathcal{F}^\bullet / H^n(\mathcal{F}^\bullet) = 0 \quad \forall n > 0 \}$$

$$D^{\geq 0} = \{ \mathcal{F}^\bullet / H^n(\mathcal{F}^\bullet) = 0 \quad \forall n < 0 \}$$

Heart $= D^{\leq 0} \cap D^{\geq 0} =$ abelian category of constructible sheaves

__but__ $D^b_c(X) \neq D^b$(cat of constructible sheaves)

i.e not ~~both~~ every complex of sheaves w/ constructible cohomology is quasi isomorphic to a ~~c~~ complex of constru-ctible sheaves.

__Question__ In the heart of a t-structure what are "kernel", "surjective" etc?

__most of an answer__: $A \to B \to C \to$

is a distinguished triangle in D w/ all 3 terms in the heart, then

$0 \to A \to B \to C \to 0$ is a short exact sequence.

__Flavor of the proof:__

__Lemma:__ The distinguished triangle in axiom 3 of t-structure is functorial.

There are __truncation functors__

$$\tau^{\leq 0} : D \longrightarrow D^{\leq 0}$$
$$\tau^{\geq 1} : D \longrightarrow D^{\geq 1}$$

and a canonical distinguished triangle

$$\tau^{\leq 0} X \longrightarrow X \longrightarrow \tau^{\geq 1} X \longrightarrow$$

__proof sketch__ Suppose we have 2 distinguished triangles as in axiom(3)

$$\begin{array}{ccccccc} A & \longrightarrow & X & \longrightarrow & B & \longrightarrow & \\ & & \| & & & & \\ A' & \longrightarrow & X & \longrightarrow & B' & \longrightarrow & \end{array}$$

Rotate 1st triangle

$$B[-1] \longrightarrow A \longrightarrow X \longrightarrow$$

Apply $\mathrm{Hom}(A', \cdot)$, to get long exact sequence:

$$\longrightarrow \mathrm{Hom}(\underset{D^{\leq 0}}{A'}, \underset{D^{\geq 2}}{B[-1]}) \longrightarrow \mathrm{Hom}(A', A) \longrightarrow \mathrm{Hom}(A', X)$$

$$\longrightarrow \mathrm{Hom}(\underset{D^{\leq 0}}{A'}, \underset{D^{\geq 0}}{B}) \longrightarrow$$

$\mathrm{Hom}(A', B[-1]) = 0$ because

axiom(1) (of t-stru)

so there is a unique $A' \to A$ such that

$$\begin{tikzcd} A \arrow[r] & X \arrow[r] \arrow[d, phantom, "\|"] & B \arrow[r] & {} \\ A' \arrow[u, dashed] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & X \arrow[r] & B' \arrow[r] & {} \end{tikzcd}$$

Repeat w/ A & A' switched to get $A \simeq A'$. Similarly $B \simeq B'$.

<u>example 1</u>

X: topological space w/ $\overbrace{\text{stratification}}^{\text{connected strata}}$

Assume all strata are even dimen-sional

$$\mathcal{D} = D^b_c(X)$$

$${}^p\mathcal{D}^{\leq 0} = \{ \mathcal{F}^\bullet \mid \dim \operatorname{supp} H^n(\mathcal{F}^\bullet) \leq -2n \}$$

equivalently $H^n(\mathcal{F}^\bullet)|_S = 0$ if $n > -\frac{1}{2}\dim S$

<u>SIDE NOTE</u>: All truncation functors $\tau^{\leq n}, \tau^{\geq n}$ commute. The composition $\tau^{\leq 0} \circ \tau^{\geq 0}: \mathcal{D} \to \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is called t-cohomology denoted ${}^tH^0 : \mathcal{D} \to \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$

In examples 0, 0' ${}^tH^0 = H^0$

$${}^{p}D^{\geq 0} = \{ \mathcal{F}^\bullet \mid \mathbb{D}\mathcal{F} \in {}^{p}D^{\leq 0} \}$$

verdier duality

Recall $\mathbb{D} = R\mathcal{H}om(\cdot, \omega_X)$

dualizing complex

if X orientable manifold then
$\omega_X = \underline{\mathbb{C}}_X[\dim X]$

<u>Thm</u> $({}^{p}D^{\leq 0}, {}^{p}D^{\geq 0})$ is a t-structure

on $D^b_c(X)$

objects in the heart are called <u>perverse</u> <u>sheaves</u>

<u>Thm</u> Let L be an irreducible local system on a stratum S. Then there exists a unique simple object in the category of perverse sheaves denoted $IC(S, L)$ such that

1) $IC(S,L)|_S \simeq L[\frac{1}{2}\dim S]$
2) support of $IC(S,L)$ is $\overline{S}$

All simple perverse sheaves arise this way.

Question: what is an $IC(S, L)$?

Approximate answer:
$H^n(IC(S,L))|_{S'}$ is a local system, so write them all down.

example 2 X = variety of 3×3 nilpotent matrices over $\mathbb{C}$. stratify by conjugacy classes = orbits for conjugation action of SL_3

3 strata:

		dim $/\mathbb{R}$
S_p =	orbit of $\begin{bmatrix} 0&1&0\\0&0&1\\0&0&0\end{bmatrix}$	12
S_m =	orbit of $\begin{bmatrix} 0&1&0\\0&0&0\\0&0&0\end{bmatrix}$	8
S_0 =	orbit of $\begin{bmatrix} 0&&\\&0&\\&&0\end{bmatrix}$	0

	equivariant local system
S_p	reps of $\mathbb{Z}/3\mathbb{Z}$, $\underline{\mathbb{C}}$, L_1, L_2
S_m	$\underline{\mathbb{C}}$
S_0	$\underline{\mathbb{C}}$

closure rels: $\bar{S}_p \supset \bar{S}_m \supset \bar{S}_0 = S_0$

Answers:

$$IC(S_p, \underline{\mathbb{C}}) = \underline{\mathbb{C}}_X[6]$$

~~for $L = L_1, L_2$~~

for $L = L_1, L_2$

$$IC(S_p, L)|_{S_p} = L[6], \quad IC(S_p, L)|_{S'} = 0$$

$$IC(S_0, \underline{\mathbb{C}}) = \underline{\mathbb{C}}_{S_0}$$

$$IC(S_m, \underline{\mathbb{C}})|_{S_m} = \underline{\mathbb{C}}_{S_m}[4],$$

$$H^{-4}(IC(S_m, \underline{\mathbb{C}}))|_{S_0} \simeq \underline{\mathbb{C}}_{S_0}$$

$$H^{-2}(IC(S_m, \underline{\mathbb{C}}))|_{S_0} \simeq \underline{\mathbb{C}}_{S_0}$$

Encode above calculations w/ polynomials

	S_p			S_m	S_0
	$\underline{\mathbb{C}}$	L_1	L_2	$\underline{\mathbb{C}}$	$\underline{\mathbb{C}}$
$IC(S_p, \underline{\mathbb{C}})$	x^{-6}			x^{-6}	x^{-6}
$IC(S_p, L_1)$		x^{-6}			
$IC(S_p, L_2)$			x^{-6}		
$IC(S_m, \underline{\mathbb{C}})$				x^{-4}	$x^{-4}+x^{-2}$
$IC(S_0, \underline{\mathbb{C}})$					x^{0}