14 Jan 2009 — Algebraic Lie Theory at the Newton Institute

P. Achar — Derived categories and perverse sheaves III

Get a local system on $X = \mathbb{C} \setminus \{0\}$ by $\mathcal{F}(U) =$ solutions of $x\dfrac{df}{dx} = \dfrac{1}{2} f(x)$ (*)

open — on U

Generalize to $\mathcal{D}$-modules:

$$\mathcal{D} = \mathbb{C}\left[x, \frac{d}{dx}\right]$$

operator which multiplies a function by x

$$\frac{d}{dx}\left(x f(x)\right) = f(x) + x\frac{df}{dx}$$

In $\mathcal{D}$: $\dfrac{d}{dx} \cdot x = x\dfrac{d}{dx} + 1$

stick to left $\mathcal{D}$-modules

Let $P = x\dfrac{d}{dx} - \dfrac{1}{2} \in \mathcal{D}$

Let

$\mathcal{O}$ = sheaf of holomorphic functions on $\mathbb{C}$

$\mathcal{D}$ acts on $\mathcal{O}(U)$ for any open set $U \subset \mathbb{C}$

A function $f(x)$ is a solution of (*)
$\iff Pf = 0$

Another way:

Let $M = \mathcal{D}/\mathcal{D}P$

consider $\mathrm{Hom}_{\mathcal{D}\text{-mod}}(M, \mathcal{O}(U))$,

M is generated as a $\mathcal{D}$-module by $1+\mathcal{D}P$
can specify a map $\mu : M \longrightarrow \mathcal{O}(U)$ by
giving $f(x) = \mu(1)$; must satisfy $Pf = 0$. so

$$\mathrm{Hom}_{\mathcal{D}\text{-mod}}(M, \mathcal{O}(U)) \longleftrightarrow \left\{ \begin{array}{l} \text{solns. of diff.} \\ \text{eqn } x\frac{df}{dx} = \frac{1}{2}f(x) \end{array} \right\}$$

can replace $\mathcal{O}(U)$ by other function spaces,
eg. $\mathbb{C}[x]$ to get polynomial solutions

M itself is "universal" solution space
to a differential equation.

Generalize:

X = complex manifold

$\mathcal{D}_X$ = sheaf of partial differential operators on X

$\mathcal{D}_X$-module = sheaf on X consisting of modules over $\mathcal{D}_X(U)$

$\mathcal{O}_X$ = sheaf of holomorphic functions on X

derived version of sheaf of solutions to a differential equation

$$RHom_{\mathcal{D}_X}(-, \mathcal{O}_X) : \mathcal{D}^b(\mathcal{D}_X) \longrightarrow \mathcal{D}^b(X)$$

($\mathcal{O}_X$ is a $\mathcal{D}_X$-module

complex of $\mathcal{D}_X$-mod

complex of sheaves of vector spaces

To get $RHom_{\mathcal{D}_X}(-, \mathcal{O}_X)$ to be well behaved

stalks of cohomology of the output should be finite dimensional. So impose a condition: "holonomic".

A holonomic $\mathcal{D}_X$-module $\rightsquigarrow$ an "overdetermined" system of PDEs

<u>Thm (Kashiwara)</u>

If M is a holonomic $\mathcal{D}_X$-module then $RHom_{\mathcal{D}_X}(M, \mathcal{O}_X)$ has finite dimensional stalks and has constructible cohomology sheaves.

So really we are looking at

$$R\mathcal{H}om_{\mathcal{D}_X\text{-mod}}(-, \mathcal{O}_X) : D^b_h(\mathcal{D}_X) \longrightarrow D^b_c(X)$$

cohomology sheaves are holono-mic $\mathcal{D}_X$-modules

cohomology sheaves are constructible

<u>Refine</u>:
Restrict to $\mathcal{D}_X$-modules w/ regular singularities.

<u>Thm</u> (Kashiwara, Beilinson-Bernstein)

$$R\mathcal{H}om(-, \mathcal{O}_X) : D^b_{rh}(\mathcal{D}_X) \xrightarrow{\sim} D^b_c(X)$$

regular holonomic

is an equivalence of categories. (Riemann-Hilbert ~~correspondence~~ correspondence)

<u>Hilbert's 21^{st} problem</u> (attributed to Riemann)

Given a specified local system, show that there exists a differential equation ~~that gives~~ whose solutions give you that local system.

In $D^b_{rh}(\mathcal{D}_X)$ have natural t-structure (abelian category of regular holonomic $\mathcal{D}_X$-modules). Follow this through the Riemann-Hilbert correspondence.

This gives the <u>perverse</u> t-structure on $D_c^b(X)$!

<u>Representation theory</u>

G - complex reductive group
$\mathfrak{g}$ - Lie algebra of $\mathfrak{g}$
$\mathfrak{g}$ - elements are tangent vectors OR left invariant vector fields on G

differential operators

Think of $U(\mathfrak{g})$ modules in the BGG category $\mathcal{O}$.

Fix an ~~central~~ regular integral central character χ. Consider only $U(\mathfrak{g})$-modules w/ this central character. So for example the principal block $\mathcal{O}_0$.

Form quotient of $U(\mathfrak{g})$ by $z = \chi(z)$

<u>Thm</u> This quotient is

$$\Gamma(\mathcal{D}^{\chi}_{G/B})$$

slight modification of $\mathcal{D}_{G/B}$ depending on χ

Thm. (Beilinson - Bernstein)

$$\left\{\begin{array}{l} U(\mathfrak{g})\text{-mods in } \mathcal{O} \\ \text{w/ central char} \\ \text{-acter } \chi \end{array}\right\} \longleftrightarrow \left\{ \mathcal{D}^{\chi}_{G/B}\text{-mod} \right\}$$

if $\chi = 0$ then $\mathcal{D}^{\chi}_{G/B} = \mathcal{D}_{G/B}$

Kazhdan - Lusztig conjecture

W = finite Weyl group of G

T_w, C_w standard and KL-basis for Hecke algebra of W.
Kazhdan - Lusztig polynomials $P_{y,w}$
$y, w \in W$ are change of basis polynomials from $C_w \to T_w$. ($P_{y,w} \in \mathbb{Z}[q, q^{-1}]$)

KL-conjecture

$$[M(w\cdot 0) : L(y\cdot 0)] = P_{y,w}(1)$$

stratify G/B by B-orbits (Bruhat deco-mposition)

orbits $X_w \longleftrightarrow w \in W$

All X_w are simply connected (in fact $X_w \cong \mathbb{A}^n$)
so no non trivial local systems

compute all $IC(X_w, \underline{\mathbb{C}})|_{X_y}$

Record the result as a polynomial

<u>THM</u> (K-L) These polynomials are $x^{-\ell(w)} P_{y,w}(x^2)$

$$
\begin{tikzcd}
\left(\begin{array}{l}\text{Hecke algebra}\\ \text{of } W\end{array}\right) \arrow[d, dash] & \\
(\text{KL-polys}) \arrow[r, dash, "\text{KL-conj}"] \arrow[d, dash, "\text{KL}"'] & \left(\begin{array}{l}\text{structure of}\\ \text{certain } U(\mathfrak{g})\text{-mods}\end{array}\right) \arrow[d, dash, "\text{BB}"] \\
\left(\begin{array}{l}\text{perverse sheaves}\\ \text{on } G/B\end{array}\right) & (\mathcal{D}_{G/B}\text{-mod}) \arrow[l, "\text{RH}"']
\end{tikzcd}
$$

Beilinson-Bernstein, Brylinski-Kashiwara.