15 Jan 2009 Algebraic Lie Theory at the Newton Institute

P. Achar Derived categories and perverse sheaves IV

The Decomposition Theorem

$f: Y \to X$ proper morphism of varieties, Y is smooth, f is semismall (condition on dim of $f^{-1}(x)$). Then
$$Rf_* \mathbb{C}_Y[\dim Y] \simeq \bigoplus \text{simple perverse sheaves}$$

X – a variety over $\overline{\mathbb{F}}_q$, defined over $\mathbb{F}_q$

$F: X \to X$, the Frobenius map ($x \to x^q$ on coordinates)

X^F = Frobenius fixed pts = $\mathbb{F}_q$ pts of X

work w/ $\overline{\mathbb{Q}}_\ell$-sheaves in ètale topology (Zariski topology has too few open sets)

Notes:

1) ètale topology is not an "ordinary" topology but a Grothendieck topology
2) "$\overline{\mathbb{Q}}_\ell$-sheaf" does not mean an assig-nment of $\overline{\mathbb{Q}}_\ell$-vector spaces to each ètale open set

BUT:

can usually ignore these complications.

eg. stalk of $\overline{\mathbb{Q}}_\ell$-sheaf is a $\overline{\mathbb{Q}}_\ell$-vector space, same for other sheaf operations

3) $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$

4) ℓ = prime $\neq$ char $\mathbb{F}_q$

$D^b_c(X)$, perverse t-structure
$P(X)$ = heart consisting of perverse sheaves
} defined as before

EXCEPT: replace $\frac{1}{2}\dim_{\mathbb{R}}$ by $\dim_{alg}$

$\mathcal{F}$: $\overline{\mathbb{Q}}_\ell$-sheaf on X, defined over $\mathbb{F}_q$
Then the Frobenius map induces an isomor-phism

$$\Phi : F^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$$

If $x \in X^F$, Φ gives an automorphism

$$\Phi_x : \mathcal{F}_x \longrightarrow \mathcal{F}_x$$

Key idea : keep ~~trag o~~ track of eigenvalues
"weights"

Def $\mathcal{F}$ (a sheaf) is pointwise pure of weight $w \in \mathbb{Z}$ if for all $x \in X^{F^n}$, all eig-envalues of Φ_x are algebraic numbers w/ absolute value $q^{nw/2}$ under any isomo-rphism $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$

eg "naive" constant sheaf $\overline{\mathbb{Q}}_{\ell_X}$ is pt. wise pure of wt. 0
can modify wts. w/ "Tate twist"

$\overline{\mathbb{Q}}_{\ell_X}(n)$ is pt. wise pure of wt. $-2n$

F^n gives $\mathbb{F}_{q^n}$-pts $\rightsquigarrow$ $\cancel{\Phi_{\mathbb{F}_q}}$ $\Phi_{\mathbb{F}_{q^n}} : \mathcal{F}_x \longrightarrow \mathcal{F}_x$, $x \in X^{F^n}$

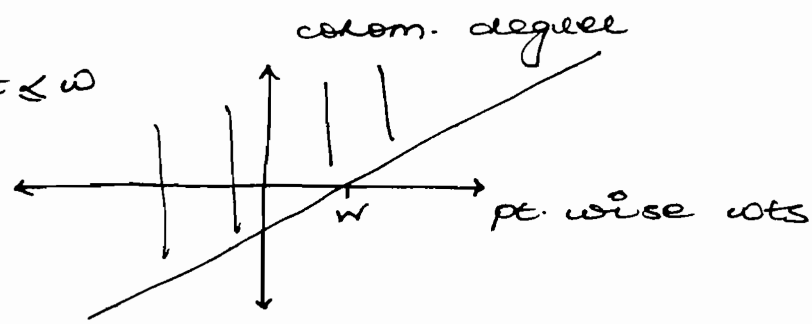
Def $\mathcal{F}$ is mixed (w/ wt $\leq \omega$) if it has a finite filtration w/ ptwise pure subquotients.

$\mathcal{D}^b_m(X)$ = full subcategory of $D^b_c(X)$ w/ objects whose cohomology sheaves are mixed

- Full triangulated subcategory
- Preserved by usual sheaf operations

Def $\mathcal{F}^\bullet \in D^b_m(X)$ has weights $\leq \omega$ if $H^k(\mathcal{F}^\bullet)$ is mixed w/ ~~ptw~~ pt. wise wts $\leq \omega + k$

obj. of wt $\leq \omega$



This category is $D_{\leq \omega}$

$\mathcal{F}^\bullet \in D^b_m(X)$ has weights $\geq \omega$ if $\mathbb{D}\mathcal{F}^\bullet \in D_{\leq -\omega}$. The category of such objects is denoted $D_{\geq \omega}$

$\mathcal{F}^\bullet$ is pure of wt ω if its in $D_{\leq \omega} \cap D_{\geq \omega}$

Aside: If X is smooth and all $H^k(\mathcal{F}^\bullet)$ are local systems,

$\mathcal{F}^\bullet$ is pure of wt ω $\iff$ $H^k(\mathcal{F}^\bullet)$ pt. wise pure of

<u>Origin</u>: Weil conjectures.
X: smooth, projective;
Expect: Frobenius should act on
$$H^k(X, \bar{\mathbb{Q}}_\ell) = H^k(R\Gamma(\bar{\mathbb{Q}}_\ell))$$
w/ eigenvalues of absolute value $q^{k/2}$

<u>Fact</u> ① (Deligne)
Description of the behaviour of
$R\mathrm{Hom}$, $\overset{L}{\otimes}$, Rf_*, f^* etc on $D_{\leq w}$, $D_{\geq w}$

<u>Cor</u> ② If $\mathcal{F}^\bullet \in D_{\leq w}$, $\mathcal{G}^\bullet \in D_{\geq w}$ and both are (*) perverse sheaves, then
$$\mathrm{Hom}(\mathcal{F}^\bullet, \mathcal{G}^\bullet[n]) = 0 \quad \text{for all } n > 0.$$

<u>Comment on proof</u>: If $\mathcal{G}^\bullet \in D_{\geq w+1}$ then holds w/o (*).
If $\mathcal{G}^\bullet \in D_{\geq w}$, $\mathcal{G} \notin D_{\geq w+1}$ (not assuming (*))
then $\mathrm{Hom}(\mathcal{F}^\bullet, \mathcal{G}^\bullet[n])$ reduces to
$$\mathrm{Hom}(\mathcal{F}^\bullet, \mathcal{G}^\bullet[-1]) = 0$$
axiom 1 of t-structure

<u>Fake proof</u>
$IC(S, L)$
suppose S open stratum, $\mathring{j}: S \hookrightarrow X$
$\mathring{j}_{!*}$ - "between" $\mathring{j}_!$ and $\mathring{j}_*$

Fact ① (above) says:
$\mathring{j}_!$ preserves $D_{\leq w}$

$\mathring{j}_*$ preserves $D_{\geq w}$
so $\mathring{j}_{!*}$ preserves both.

<u>Thm</u> Every pure perverse sheaf is semi-simple.

<u>proof</u> $\mathcal{F}^\bullet$ - a pure perverse sheaf
Let $K^\bullet \subset \mathcal{F}^\bullet$ be sum of all simple sub perverse sheaves of $\mathcal{F}^\bullet$
= maximal semisimple subperverse sheaf
Form short exact sequence:

$$0 \to K^\bullet \to \mathcal{F}^\bullet \to \mathcal{G}^\bullet \to 0$$

OR distinguished triangle

$$K^\bullet \to \mathcal{F}^\bullet \to \mathcal{G}^\bullet \to K^\bullet[1]$$

<u>Fact</u>: subquotients of perverse sheaves respect wts. So $\mathcal{G}^\bullet, K^\bullet$ are pure of the same weight as $\mathcal{F}^\bullet$.

cor ② says $\mathrm{Hom}(\mathcal{G}^\bullet, K^\bullet[1]) = 0$.
So the distinguished triangle splits:

$$\mathcal{F}^\bullet \simeq K^\bullet \oplus \mathcal{G}^\bullet.$$

contradiction: $\mathcal{G}^\bullet$ contains simple subobject not in $K^\bullet$.

<u>Thm</u> (weight filtration)
Every perverse sheaf admits a canonical finite filtration

$$\cdots \subset \mathcal{F}_{w-1} \subset \mathcal{F}_w \subset \mathcal{F}_{w+1} \subset \cdots$$

s.t. $\mathcal{F}_w / \mathcal{F}_{w-1}$ is pure of wt. w

<u>proof</u> (exercise)

<u>proof of Decomposition Theorem</u>

a) by fact ①, Rf_* preserves both $\Delta_{\leq w}$ and $\Delta_{\geq w}$, so takes pure objects to pure objects

b) $\overline{\mathbb{Q}}_{\ell_Y}$ [dim Y] for Y smooth is a pure perverse sheaf (using Poincaré duality). In fact it is simple.

c) f semismall: Rf$_*$ takes ~~perverse sheaves~~ $\overline{\mathbb{Q}}_{\ell_Y}$ [dim Y] to perverse sheaves

combine:

Rf$_*$ $\overline{\mathbb{Q}}_{\ell_Y}$ [dim Y] is a pure perverse sheaf on X, so it is semisimple.

~~plot~~
~~As~~ Side note:

ordinary topology: declare certain subsets of X to be open

Grothendieck topology: declare certain collec-tions of maps to be ~~over~~ open covers

$\uparrow$

ètale topology : these are the ètale maps.

ètale = alg. geom version of covering map

eg. local system L on $X = \mathbb{C} \setminus \{0\}$

L ~~sheaf~~ = solns to $x \frac{d}{dx} f(x) = \frac{1}{2} f(x)$.

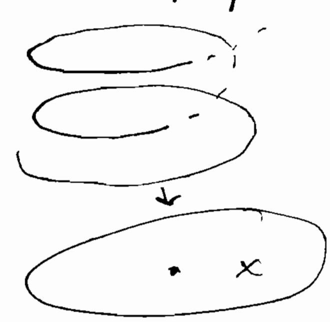
classical top. : $L(U) = \begin{cases} 0 & \text{if } U \text{ wraps around } 0 \\ \mathbb{C} x^{1/2} & \text{o.w} \end{cases}$

$\uparrow$ connected

Zariski top. : $L = 0$

An ètale map : $\phi : Y \longrightarrow X$, $\phi(x) = x^2$

$Y = \mathbb{C} \setminus \{0\}$



$L(U) = \mathbb{C}\{\text{sections of } \phi\}$

Rough idea: $L(X) = 0$. "On the ètale open set Y, ϕ has a section namely $id : Y \to Y$."

$L(Y) \cong \mathbb{C}\{id\}$