16 Jan 2009 Algebraic Lie Theory at the Newton Institute

P. Achar Derived categories and perverse sheaves V

Topics:
1) Purity and pt. wise purity
2) Springer correspondence
3) Intersection cohomology
4) Setting of coherent sheaves

1) Pt. wise purity: a strong condition on single sheaf

Purity: a condition on chain complexes, not neccesarily related to pt. wise purity

Def $\mathcal{F}^\bullet \in D^b_m(X)$ is ~~pure~~ pt. wise pure of wt. w if $H^k(\mathcal{F}^\bullet)$ is pt. wise pure of wt. $w+k$.
Last lecture: for local systems on smooth X:

pure $\Longleftrightarrow$ pt. wise pure

In general neither implies the other.
Thm (Kazhdan - Lusztig)
Simple perverse sheaves on G/B and on the nilpotent variety $\mathcal{N}$ are both pure and pt.-wise pure.
$\hookrightarrow$ Major consequences for computing IC's.

2) Springer correspondence
$\mathcal{N}$ = ~~the~~ nilpotent variety in Lie algebra $\mathfrak{g}$ of a complex reductive algebraic group G

$$\tilde{\mathcal N} = \{(x,b) \mid x \in \mathcal N,\ b \in \text{Borel subalgebra of } \mathfrak g \text{ containing } x\}$$

$$\begin{tikzcd} \tilde{\mathcal N} \arrow[d, "\pi"] \\ \mathcal N \end{tikzcd}$$

$\tilde{\mathcal N}$ is smooth, π is ~~pt-wise~~ semi-small and proper.

Invoke the decomposition Thm

$$R\pi_* \underline{\mathbb C}_{\tilde{\mathcal N}}[\dim \mathcal N] = \bigoplus IC(S,L) \otimes \underbrace{V_{S,L}}$$

$$\dim V_{S,L} = [R\pi_* \underline{\mathbb C}_{\tilde{\mathcal N}}[\dim \mathcal N] : IC(S,L)]$$

<u>Thm (Borho–Macpherson)</u>

- Each (non-zero) $V_{S,L}$ carries irreducible W representations
- Every irred. W-rep occurs exactly once in the direct sum

so we get $\mathrm{Irr}(W) \longleftrightarrow \{(S,L)\}$
(springer correspondence)

3) Intersection cohomology:

1st lecture:

(compact orie-ntable manif-old X)

$$\begin{tikzcd}[column sep=small] & H^n(R\Gamma(\mathbb D \underline{\mathbb C}_X)) \arrow[dl, "\mathbb D\underline{\mathbb C}_X = \underline{\mathbb C}_X[\dim X]"'] \arrow[dr, "R\Gamma \circ \mathbb D = \mathbb D \circ R\Gamma"] & \\ H^{n+\dim X}(X, \mathbb C) & & H^{-n}(X;\mathbb C)^* \arrow[ll, "\sim"', "\text{classical poincare duality}"] \end{tikzcd}$$

Perverse t-structure

$${}^pD^{\le 0} = \{\dots\},\quad {}^pD^{\ge 0} = \mathbb D({}^pD^{\le 0}),\ \text{so}$$

$P(X) = {}^pD^{\le 0} \cap {}^pD^{\ge 0}$ is stable under $\mathbb D$

so $\mathbb D$ induces an anti-autoequivalence

$\mathbb D : P(X) \longrightarrow P(X)$

$\mathbb{D}$ must take simple objects to simple objects

$$\mathbb{D}\, IC(S, L) = IC(S, L^\vee)$$

$L^\vee$ — dual local system to L

special case:

$$\mathbb{D}\, IC(S, \underline{\mathbb{C}}_S) \simeq IC(S, \underline{\mathbb{C}}_S)$$

Assume that S is a dense open stratum in X.

supp. $IC(S, \underline{\mathbb{C}}_S)$ is all of X.

Let $\widetilde{IC} = IC(S, \underline{\mathbb{C}}_S)[-\tfrac{1}{2}\dim_{\mathbb{R}} S]$

$$\begin{tikzcd}
 & H_n(R\Gamma(\widetilde{IC})) \arrow[dl, "\mathbb{D}\widetilde{IC} = \widetilde{IC}[\dim X]"'] \arrow[dr, "R\Gamma\circ\mathbb{D} = \mathbb{D}\circ R\Gamma"] & \\
H^{n+\dim X}(X, \widetilde{IC}) & & H^{-n}(X, \widetilde{IC})^* \arrow[ll, "\sim"']
\end{tikzcd}$$

↑ Goresky - MacPherson: Poincaré duality for singular spaces.

$H^k(X; \widetilde{IC}) = IH^k(X) =$ intersection cohomology groups.

4) In the setting of coherent sheaves (Deligne, Bezrukavnikov)

X - variety / algebraically closed field

For now fix a stratification

$Coh(X) =$ category of coherent sheaves on X

$\mathcal{D}^b(X) = D^b(Coh(X))$

$\mathcal{O}(X) =$ structure sheaf

Analogue of Poincaré-Verdier duality $\mathbb{D}_{PV}$	:	Serre Grothendieck duality $\mathbb{D}_{SG}$
local system (locally free)		vector bundle (loc. free sheaf over

Goal: Find self-dual t-structure on $\mathcal{D}^b(X)$
(*) the subcategory of objects whose cohomol-ogy are vector bundles along strata.

Problems:

1) Groth.-Serre duality involves $\dim_{alg}$ not $\dim_{\mathbb{R}}$. Need strata to have even alg. dim for self duality
2) Restriction of coherent sheaves to strata is not an exact functor (must $\otimes$). Must do derived restricti-on. Restriction and H^* don't commute.
3) (*) is not triangulated!

Assume an algebraic group ~~acts~~ G acts on X w/ finitely many orbits. Stratify by orbits. Work in $\mathcal{D}^G(X) = \underbrace{D^b(\mathrm{Coh}^G(X))}_{G\text{-equivariant coherent sheaves}}$

G-equivariant coherent sheaves on 1 orbit is automatically ~~free~~ locally free.
Now (*) makes sense. Get a t-structure:
Thm ① Let $\mathcal{P}(X)$ = category of G-equivariant perverse coherent sheaves.

1) Every object has finite length
2) The simple objects are $\mathcal{IC}(S, \mathcal{L})$, $\mathcal{L}$ irred. G-equivariant vector bundle on S
3) $\mathbb{D}_{SG}\,\mathcal{IC}(S, \mathcal{L}) \simeq \mathcal{IC}(S, \mathcal{L}^\vee)$

Key example

$X = \mathcal{N} \circlearrowleft G$

Bezrukavnikov: used perverse cohomology sheaves on $\mathcal{N}$ to prove a conjecture of Lusztig-Vogan on $K_G(\mathcal{N})$

Staggered sheaves (A.; A.-Treumann)

Idea: use information from G-action to impose a "filtration" on perv. coh. sh. construction. Get a new t-structure on $\mathcal{D}^G(X)$. $\mathcal{M}(X)$ = heart of this t-structure "staggered sheaves"

Thm (A.; A.-Treumann) In $\mathcal{M}(X)$

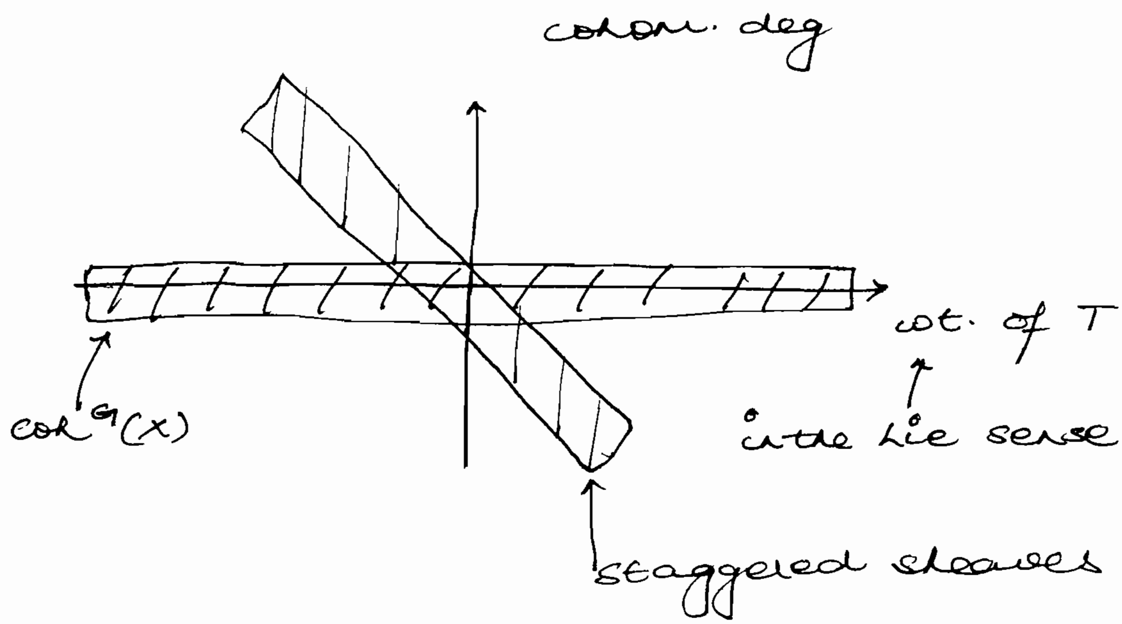
1) Every object has finite length
2) The simple objects are $\mathcal{IC}(S, \mathcal{L})$
3) $\mathbb{D}_{SG}\, \mathcal{IC}(S, \mathcal{L}) \simeq \mathcal{IC}(S, \mathcal{L}^\vee)$
4) orbits need not have even dimension
5) Get a "weight structure" on $\mathcal{D}^G(X)$ s.t.
   a) Every simple staggered sheaf is pure
   b) Every pure object is semisimple
6) $\mathcal{M}(X)$ has enough projectives and injectives
7) $\mathcal{M}(X)$ has a collection of "standard objects" (analogues of Verma modules) and "costandard objects".

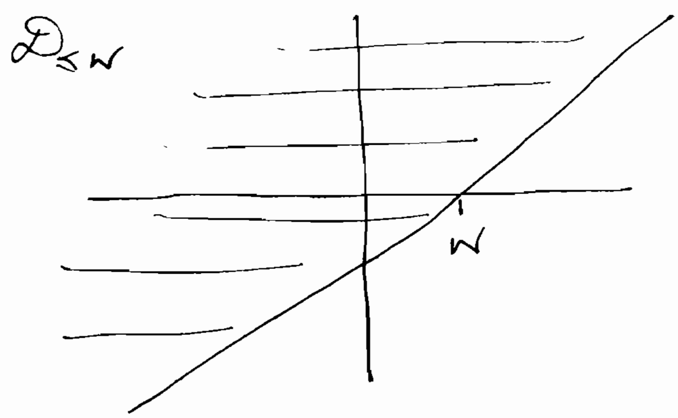
side note: "weight structure" on $\mathcal{D}^G(X)$
for staggered sheaves: staggered t-structure depends on choices. choose a filtration of each $\mathrm{Coh}^G(S)$ ← orbit subject to some axioms

$X = pt$; $G = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \subset SL_2$

$\mathrm{Coh}^G(X) = \mathrm{Rep}(G)$; $\mathrm{Irr}(G) \simeq \mathbb{Z}$

$T \subset G$.





$\mathcal{D}_{\geq w} = \mathbb{D}_{SG}(\mathcal{D}_{\leq -w})$