12 Jan 2009 Algebraic Lie Theory at the Newton Institute

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Plan:

I. Motivation from representation theory of finite groups
II. sl_2-categorification and construction of derived equivalences
III. Perverse equivalences.

Broué's abelian defect group conjecture

G - finite group
k - algebraically closed field
k[G] - group algebra

$\mathbb{C}[G]$ is semisimple so
$$\mathbb{C}[G] \simeq M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_s}(\mathbb{C})$$

$M_n(\mathbb{C}) = n \times n$ matrices over $\mathbb{C}$

Assume char $k = p > 0$

$k[G] \simeq B_1 \times \cdots \times B_s$, B_i are the blocks of $k[G]$.

$k[G]$-mod $\simeq B_1$-mod $\oplus \cdots \oplus B_s$-mod

<u>Example</u> $G = S_3$

$p>3$: $k[G] = M_2(k) \times k \times k$

$p=2$: $k[G] = M_2(k) \times k[x]/(x^2)$

$p=3$: $k[G]$ is indecomposable

A <u>defect group</u> of a block B is a subgroup $\Delta \leq G$ such that every B-module is a summand of a module induced from Δ and is minimal w/ this property.

Δ is a ~~sub~~ p-subgroup of G, unique upto conjugacy.
$\uparrow$ char k

$\Delta = 1 \iff B$ is a matrix algebra

Δ is cyclic $\iff$ There are finitely many indecomposable B-modules (upto $\cong$)

Each block B has a Brauer correspondent C a block of $k[N_G(\Delta)]$
$\uparrow$ defect group of B.

<u>Conjecture</u> (Alperin) If Δ is abelian then B and C have the same # of simple modules.

<u>example</u> $G = A_5$, $p = 3$

$k[G] \simeq B \times M_3(k) \times M_3(k)$

B has defect group $D \simeq \mathbb{Z}/3\mathbb{Z}$
and $N_G(D) \simeq S_3$.
So its Brauer correspondent is
$C = k[S_3]$.

Both B and C have 2 simple modules
Explanation: B and C are Morita equivalent, i.e. $B\text{-mod} \simeq C\text{-mod}$

<u>example</u> $G = A_5$, $p = 2$

$k[G] \simeq B \times M_4(k)$

B has defect group $D \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

w/ Brauer correspondent $k[N_G(D)] \simeq k[A_4]$

Both B and C have 3 simple modules
But $B\text{-mod} \not\simeq C\text{-mod}$.
Instead, & Ricard showed that

$$D^b(B\text{-mod}) \xrightarrow{\sim} D^b(C\text{-mod})$$

$$\text{simple objects of } B\text{-mod} \left\{ \begin{array}{lcl} S_1 & \longmapsto & T_1 \\ S_2 & \longmapsto & \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \\ S_3 & \longmapsto & \begin{pmatrix} T_1 \\ T_3 \end{pmatrix} [1] \end{array} \right.$$

$\{T_i\text{'s simples in } C\text{-mod}$

[1] element of $\mathrm{Ext}^1(T_1, T_2)$

<u>Conjecture (Broué)</u> If $\mathcal{D}$ is abelian B and C are derived equivalent, i.e
$$\mathcal{D}^b(B\text{-mod}) \simeq \mathcal{D}^b(C\text{-mod})$$

Recall that given an abelian category $\mathcal{A}$, the <u>Grothendieck group</u> $K_0(A)$ is the free abelian group on symbols $[X]$, $X \in A$ modulo
$$[Y] = [X] + [Z] \quad \text{for each}$$
short exact sequence
$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

If $\mathcal{A}$ is Artinian (i.e finite composition series) then $[L]$, L-simple is a basis of $K_0(\mathcal{A})$.

$$K_0(B\text{-mod}) = \bigoplus_{S\text{-simple}} \mathbb{Z}[S]$$

If $F: \mathcal{A} \longrightarrow \mathcal{B}$ is a exact functor between abelian categories then

$$[F]: K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{B})$$
$$[M] \longmapsto [FM]$$
is a homomorphism. If F is an equivalence, we get an isomorphism in K_0 preserving simples.

$\mathcal{T}$ - triangulated category
Define $K_0(\mathcal{T})$ as in abelian case by replacing short exact sequences w/ ~~exa~~ distinguished triangles.

$\mathcal{A}$ - abelian

$$\mathcal{A} \hookrightarrow D^b(\mathcal{A})$$

induces $K_0(\mathcal{A}) \xrightarrow{\sim} K_0(D^b(\mathcal{A}))$

For $X^\bullet \in D^b(\mathcal{A})$ in $K_0(D^b(\mathcal{A})$

$$[X^\bullet] = \sum_i (-1)^{i-1} H^i(X^\bullet)$$

depend on
$\mathcal{A} \hookrightarrow D^b(\mathcal{A})$
being in degr
-ee 1.}

so $D^b(B\text{-mod}) \xrightarrow{\sim} D^b(C\text{-mod})$
$\Rightarrow$ $K_0(B\text{-Mod}) \xrightarrow{\sim} K_0(C\text{-mod})$

but classes of simples may not be preserved.

<u>Symmetric groups</u>

B: block of $k[S_n]$
B': block of $k[S_{n'}]$

D, D': respective defect groups
C, C': resp. Brauer correspondents.

<u>Fact</u>: $D \simeq D' \Leftrightarrow C$ and C' are Morita equivalent

given B w/ abelian D, Broué's conjecture can be proved for some B' w/ $D' \simeq D$.
(Rickard, Marcus, C-Kessar)
B' – Rouquier block.

Broué's abelian defect group conjecture for symmetric groups follows from:

<u>Thm</u> (C-Rouquier)
B and B' are derived equivalent
$\iff D \simeq D'$.

<u>Lascoux-Leclerc-Thibon's approach</u>

$M \in k[S_n]\text{-mod}$ $\qquad$ $\mathcal{F} = \bigoplus_{n \geq 0} k[S_n]\text{-mod}$

$\mathrm{Res}^{S_n}_{S_{n-1}}(M)$ defines an endofunctor

$\mathrm{Res} : \mathcal{F} \to \mathcal{F}$. Similarly, define
$\mathrm{Ind} : \mathcal{F} \to \mathcal{F}$

$L_n \in \mathrm{End}(k[S_n])^{kS_{n-1}}$
$L_n = (1,n) + (2,n) + \cdots + (n-1,n)$

$\cancel{E_a M = \{ m \in M \mid (L_n - a)^N m = 0 \text{ for } N \gg 0 \}}$

$E_a M = \{ m \in \cancel{M}\ \cancel{Re}\ \mathrm{Res}\, M \mid (L_n - a)^N m = 0, \text{ for } N \gg 0 \}$

$F_a M = \{ m \in \mathrm{Ind}\, M \mid (L_n - a)^N m = 0, \text{ for } N \gg 0 \}$.

$M \in S_n\text{-mod}$

<u>Thm</u> The action of $e_a = [E_a]$ and $f_a = [F_a]$ on $K_0(\mathcal{F})$ q extends to an action of $\widehat{sl}_p$, w/ weight space decomposition

$$K_0(\mathcal{F}) = \bigoplus_{\substack{B\text{-blocks}\\ \text{of symmet}\\ \text{-ric groups}}} K_0(B)$$

and the Weyl group W acts on wt. spaces (= blocks) transitively on blocks of a fixed defect group