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J. Chuang sl$_2$-categorifications and derived equivalences II

categorification of sl$_2$

sl$_2(\mathbb{C})$ has basis $e=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $f=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$.

V - finite dimensional sl$_2$-module

$V=\bigoplus_{\lambda\in\mathbb{Z}} V_\lambda$, $V_\lambda=\{v\in V \mid hv=\lambda v\}$

$$\begin{tikzcd}\cdots \arrow[r, "e"] & V_{\lambda-2} \arrow[r, "e"] \arrow[l, bend left, "f"] & V_\lambda \arrow[r, "e"] \arrow[l, bend left, "f"] & V_{\lambda+2} \arrow[r, "e"] \arrow[l, bend left, "f"] & \cdots\end{tikzcd}$$

integrates to an action of $SL_2(\mathbb{C})$

$\Theta\in SL_2(\mathbb{C})$ given by

$$\Theta=\begin{pmatrix}0&1\\-1&0\end{pmatrix}=\exp(-f)\exp(e)\exp(-f)$$

$$\Theta : V_\lambda \xrightarrow{\sim} V_{-\lambda}$$

Now categorify:

V: Grothendieck group (complexified) $K_{\mathbb{C}} := \mathbb{C}\otimes_{\mathbb{Z}} K_0(A)$ of an abelian category A.

e, f : classes of exact functors $E, F : A \longrightarrow A$
Assume E is left and right adjoint to F.

Weight space decomposition $A = \bigoplus_{\lambda \in \mathbb{Z}} A_\lambda$,

$K_{\mathbb{C}}(A_\lambda) = V_\lambda$

The reflection ~~means~~ ~~Θ comes from~~ Θ comes from $\Theta : D^b(A_{-\lambda}) \xrightarrow{\sim} D^b(A_\lambda)$.
To prove that Θ exists, need an action of the affine Hecke algebra on E^n and F^n.

<u>Motivation</u>: systematic construction of interested derived equivalences, eg. for Broué's abelian defect group conjecture.

<u>examples</u> let k be a field.

~~1) $k\text{-mod} \underset{F=id}{\overset{E=id}{\rightleftarrows}} k\text{-mod}$ $\mathbb{C} \rightleftarrows \mathbb{C}$~~

~~1) $k\text{-mod} \longrightarrow k\text{-mod}$~~

1) $$A = \underbrace{k\text{-mod}}_{V_1} \oplus \underbrace{k\text{-mod}}_{V_2}.$$

$$\begin{tikzcd} V_1 \arrow[r, shift left, "E"] & V_2 \arrow[l, shift left, "F"] \end{tikzcd}$$

~~Θ~~

$$\begin{tikzcd} \mathbb{C}_{-1} \arrow[r, shift left, "e"] & \mathbb{C}_1 \arrow[l, shift left, "f"] \end{tikzcd}$$

~~A~~

2)'
$$\begin{tikzcd} k\text{-mod} \arrow[r, "E"] & k\text{-mod} \arrow[r, "E"] \arrow[l, bend left, "F"] & k\text{-mod} \arrow[l, bend left, "F"] \end{tikzcd}$$

$$\underbrace{E : V \longrightarrow V^{\oplus m}}_{\text{DOESN'T WORK.}}$$

Fix:

2)
$$\begin{tikzcd} k\text{-mod} \arrow[r, "E = \mathrm{Ind}"] & \frac{k[x]}{x^2}\text{-mod} \arrow[l, bend left, "F = \mathrm{Res}"] \arrow[r, "E = \mathrm{Res}"] & k\text{-mod} \arrow[l, bend left, "F = \mathrm{Ind}"] \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{C}_{-2} \arrow[r, "e"] & \mathbb{C}_0 \arrow[l, bend left, "f"] \arrow[r, "e"] & \mathbb{C}_2 \arrow[l, bend left, "f"] \end{tikzcd}$$

3) Interchange k and $\frac{k[x]}{x^2}$ in 2)

4) Let $B = k[x]/x^3$

$$\begin{tikzcd} k\text{-mod} \arrow[r, "\mathrm{Ind}"] & B\text{-mod} \arrow[l, bend left, "\mathrm{Res}"] \arrow[r, "Z \otimes_B -"] & B\text{-mod} \arrow[l, bend left, "Z \otimes_B -"] \arrow[r, "\mathrm{Res}"] & k\text{-mod} \arrow[l, bend left, "\mathrm{Ind}"] \end{tikzcd}$$

$$Z = \ker\left(B \otimes B \xrightarrow{\text{mult.}} B\right)$$

5) char $k = 2$.

$$\begin{tikzcd} B(k[S_3])\text{-mod} \arrow[r, "\mathrm{Ind}"] & B(k[S_4])\text{-mod} \arrow[l, bend left, "\mathrm{Res}"] \arrow[r, "\mathrm{Ind}"] & B(k[S_5])\text{-mod} \arrow[l, bend left, "\mathrm{Res}"] \arrow[r, "\mathrm{Ind}"] & B(k[S_6])\text{-mod} \arrow[l, bend left, "\mathrm{Res}"] \end{tikzcd}$$

certain simple blocks

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\begin{pmatrix} 3 \\ 2 \end{pmatrix}"] & \mathbb{C}^2 \arrow[l, bend left, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] \arrow[r, "\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}"] & \mathbb{C}^2 \arrow[l, bend left, "\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}"] \arrow[r, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] & \mathbb{C} \arrow[l, bend left, "\begin{pmatrix} 3 \\ 4 \end{pmatrix}"] \end{tikzcd}$$

extension of example 4 by example 1.

6) $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$; $\mathfrak{h} = \{\text{diagonal matrices}\}$
$\mathfrak{n} = \{\text{upper triangular matrices}\}$

$\mathcal{A}$ = category $\mathcal{O}$ for $\mathfrak{g}$: f.g $\mathfrak{g}$-modules,
$\mathfrak{h}$-diagonalizable,
locally $\mathfrak{n}$-finite.

$M \in \mathcal{A}$ $\qquad \mathfrak{gl}_n \otimes M \longrightarrow M$

$W = \mathbb{C}^n$ (natural representation)

$$W^* \otimes W \otimes M \longrightarrow M$$

$$\rightsquigarrow \quad L_M : W \otimes M \longrightarrow W \otimes M$$

$E_a M = \{ v \in W \otimes M \,/\, (L_M - a)^N v = 0,\ N \gg 0 \}$

$W \otimes M = \bigoplus_{a \in \mathbb{C}} E_a M$; $\qquad W^* \otimes M = \bigoplus_{a \in \mathbb{C}} F_a M$

Then $e = [E_a]$, $f = [F_a]$ give an action of $\mathfrak{sl}_2(\mathbb{C})$ on $K_{\mathbb{C}}(\mathcal{A})$.

By construction, the functor $E_a : \mathcal{A} \longrightarrow \mathcal{A}$ comes w/ $X : E_a \longrightarrow E_a$ where $X_M = L_M - a$
Consider

$T : W \otimes W \otimes M \longrightarrow W \otimes W \otimes M$
$\qquad w \otimes w' \otimes m \longmapsto w' \otimes w \otimes m$

<u>Exercise (Arakawa-Suzuki)</u>

$$\begin{tikzcd}
W \otimes W \otimes M \arrow[d, "\mathrm{id} \otimes L_M"] \arrow[d, bend right=60, "L_{W\otimes M} \circ T"'] \arrow[d, bend left=60, "T_M \circ (\mathrm{id}\, L_M) + \mathrm{id}"] \\
W \otimes W \otimes M
\end{tikzcd}$$

It follows that T acting on $V\otimes V\otimes -$ restricts to an endomorphism of E_a^2

$\underline{sl_2}$-categorification

k-linear abelian category $\mathcal{A}$ w/ finite composition series together w/ exact $E, F : \mathcal{A} \longrightarrow \mathcal{A}$ s.t

- E is left and right adjoint to F
- Action of $[E]$ and $[F]$ on $V = K_{\mathbb{C}}(\mathcal{A})$ induces locally finite action of sl_2
- $\mathcal{A} = \bigoplus_{\lambda \in \mathbb{Z}} \mathcal{A}_\lambda$, $K_{\mathbb{C}}(\mathcal{A}_\lambda) = V_\lambda$

and natural transformations $X : E \longrightarrow E$ $T : EE \longrightarrow EE$, s.t

$$T^2 = id_{EE} \quad \text{and}$$

$$(T id_E)\cdot(id_E T)\cdot(T id_E) = (id_E T)\cdot(T id_E)\cdot(id_E T) \quad \text{in } End(E^3)$$

- $T\cdot(id_E X) = (X id_E)\circ T - id_E$
- $X_M \in End(E(M))$ is ~~locally~~ ~~locally~~ nilpotent.

Theorem The action of $\Theta \in End\, V$ lifts to ~~an equi~~ a self ~~&~~ equivalence of
$\Theta : D^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{A})$ restricting to
$D^b(\mathcal{A}_{-\lambda}) \xrightarrow{\sim} D^b(\mathcal{A}_\lambda)$.

(construction of Θ is due to Rickard)

Link to (degenerate) affine Hecke algebra

$$H_n^{aff} = k\langle T_1, \dots, T_{n-1}, x_1, \dots, x_n \rangle / \sim$$

$$\underset{\substack{\uparrow \\ \text{as vector spaces}}}{=} k[S_n] \otimes k[x_1, \dots, x_n]$$

Get homomorphism

$$H_n^{aff} \longrightarrow End(E^n)$$

T_i = swap at the ith tensor.

x_i = action of JM element

$$T_i x_{i+1} = x_i T_i - 1$$

$$0 = \sum_{i,j,k} \frac{(-1)^{i+k}}{i!\,j!\,k!} f^i e^j f^k$$