16 Jan 2009   Algebraic Lie Theory at the Newton Institute

J. Chuang sl_2-categorifications and derived equivalences III

sl_2-categorification:

- $E, F : \mathcal{A} \to \mathcal{A}$ exact functors, biadjoint descend to sl_2 action on $K_0(\mathcal{A})$.
- $\mathcal{A} = \bigoplus_{\lambda \in \mathbb{Z}} \mathcal{A}_\lambda$ where $K_0(\mathcal{A}_\lambda) = K_0(\mathcal{A})_\lambda$.
- compatible actions of H_n^{aff} on E^n
- Goal: lift $\theta = \sum_{i,j,k \geq 0} (-1)^{i+k} f^{(i)} e^{(j)} f^{(k)}$

on $K_0(\mathcal{A})$ to $\Theta : D^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{A})$.

Remark 1) H_n^{aff} is either affine degenerate or non-degenerate affine Hecke algebra. Even nilHecke algebra (Rouquier).
2) Cautis-Kamnitzer-Licata have version for triangulated categories.

What is $E^n/n!$?

$c_n = \sum_{w \in S_n} w \in k[S_n] \subseteq H_n^{aff}$ $E^{(n)} := c_n E^n \subseteq E^n$

$c_{-n} = \sum_{w \in S_n} \mathrm{sgn}(w) w \in k[S_n] \subseteq H_n^{aff}$ $E^{-(n)} := c_{-n} E^n \subseteq E^n$

Prop $E^n \simeq n! E^{(n)} \simeq n! E^{-(n)}$

proof $\bigoplus_{0 \leq i_j \leq n-j} x_1^{i_1} \cdots x_n^{i_n} E^{(n)} \simeq E^n$

and each summand is isomorphic to $E^{(n)}$ using the representation theory of H_n^{aff}

Fix adjunction $(E, F) \rightsquigarrow \mathrm{End}(E^n) \simeq \mathrm{End}(F^n)^{op}$
$\rightsquigarrow$ action of H_n^{aff} on F^n

Remark : Recall : $\cancel{\frac{k[x]}{x^2}}$

$$\frac{k[x]}{x^2}\text{-mod} \underset{\text{Ind}}{\overset{\text{Res}}{\rightleftarrows}} k\text{-mod} \underset{\text{Res}}{\overset{\text{Ind}}{\rightleftarrows}} \frac{k[x]}{x^2}\text{-mod}$$

not a ~~(Strong)~~ ~~sl~~ sl_2-categorification.
(It is a weak - sl_2 - categorification)
E^2 is indecomposable!

Rickard's complex Θ

$$\Theta|_{V_{-\lambda}} = \sum_{d\geq 0} (-1)^d e^{(\lambda+d)} f^{(d)} : V_{-\lambda} \longrightarrow V_\lambda$$

define : $\Theta(\lambda) : D^b(\mathcal{A}_{-\lambda}) \longrightarrow D^b(\mathcal{A}_\lambda)$
by $\Theta(\lambda)_d \xrightarrow{\partial} \Theta(\lambda)_{d-1}$

$$\begin{array}{ccc} \Theta(\lambda)_d & \xrightarrow{\partial} & \Theta(\lambda)_{d-1} \\ \| & & \| \\ E^{(\lambda+d)}F^{(d)} & & E^{(\lambda+d-1)}F^{(d-1)} \\ \cap & & \cap \\ E^{\lambda+d}F^d & & E^{\lambda+d-1}F^{d-1} \end{array}$$

$\| \; E^{\lambda+d-1}EFF^{d-1} \nearrow id_{E^{\lambda+d-1}} \circ \varepsilon \circ id_{F^{d-1}}$

counit of adjunction

ε needs to be ~~coherently~~ compatibly chosen with
$(E,F)\; 2 \rightsquigarrow \text{End}(E)^\wedge \xrightarrow{\sim} \text{End}(F)^{op}$

Thm A
- $\Theta(\lambda)$ is a complex because ($c_2 \cdot c_2 = 0$).
- $\Theta(\lambda) : D^b(\mathcal{A}_{-\lambda}) \xrightarrow{\sim} D^b(\mathcal{A}_\lambda)$
- $[\Theta] = \Theta$, where $\Theta = \bigoplus_\lambda \Theta(\lambda)$

proof (sketch)

$V = \bigoplus_{n\geq 0} V(n)$ (= sum of irred submodules of dim $n+1$)

does not lift to

$\mathcal{A} = \bigoplus \mathcal{A}(n)$ (example in prev. lecture)

Instead $0 \subseteq V(\leq 0) \subseteq V(\leq 1) \subseteq \cdots \subseteq V$,
$V(\leq n) = \bigoplus_{i \leq n} V(i)$ does lift

$0 \subseteq \mathcal{A}(\leq 0) \subseteq \mathcal{A}(\leq 1) \subseteq \cdots \subseteq \mathcal{A}$

$\mathcal{A}(\leq n) = \{ M \in \mathcal{A} \mid [M] \in V(\leq n) \}$

<u>sub</u>-<u>prop</u>. $\mathcal{A}(\leq n)$ is a Serre subcategory, i.e closed under kernels, cokernels and extensions.

$$\mathcal{A}((n)) := \mathcal{A}(\leq n) \big/ \mathcal{A}(\leq n-1)$$

sl_2-categorification on $\mathcal{A}$ restricts to one on $\mathcal{A}(\leq n)$ and passes to $\mathcal{A}((n))$

This reduces to isotypic case.

<u>sub-Thm</u> Suppose $V = K_{\mathbb{C}}(\mathcal{A})$ is the sum of irred. sl_2-mods of dim $n+1$. Then $\Theta(\lambda)$ has cohomology concentrated in degree $m = \frac{n-\lambda}{2}$ and $H^m(\Theta(\lambda)) : \mathcal{A}_{-\lambda} \xrightarrow{\sim} \mathcal{A}_\lambda$ and

$$\Theta(\lambda) : D^b(\mathcal{A}_{-\lambda}) \xrightarrow{\sim} D^b(\mathcal{A}_\lambda) \qquad \text{"shift by } m\text{"}$$

can make further reduction to case: V is <u>irreducible</u> of dim $n+1$ and $\mathcal{A}_n$ is semisimple. Then there is a unique such sl_2-categorification. can be constructed as follows

$$\begin{tikzcd}
H_i^{aff} \arrow[r, hook] \arrow[d] & H_n^{aff} \arrow[d] \\
\bar{H}^{\circ}_{i,n} \arrow[r, hook] & H_n^{aff} \big/ \langle k[x_1, \ldots, x_n]_+^{S_n} \rangle
\end{tikzcd}$$

simple algebra

finite dimensional symmetric algebra w/ unique simple.

~~[illegible]~~

$\mathcal{A} = \bigoplus_{0 \le i \le n} \overline{H}_{i,n}\text{-mod}$, $E = \mathrm{Ind}$, $F = \cancel{\ }\mathrm{Res}$
is an $\mathfrak{sl}_2$-categorification in which
X and T "are ~~obv~~ obvious".

<u>Aside</u>:

$$\overline{H}_{i,n} = \mathrm{Mat}_{i!}(H^*(Gr(i;n)))$$

<u>Defn</u> $\mathcal{A}, \mathcal{B}$ abelian categories. An equivalence
$F : D^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{B})$ is <u>perverse</u> if there
exist $0 = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_r = \mathcal{A}$
$0 = \mathcal{B}_0 \subseteq \mathcal{B}_1 \subseteq \cdots \subseteq \mathcal{B}_r = \mathcal{B}$ and
$p : \{1, \ldots, r\} \longrightarrow \mathbb{Z}$ s.t. $\mathcal{A}_i, \mathcal{B}_i$ are Serre
subcategories and

- F restricts to $D^b_{\mathcal{A}_i}(\mathcal{A}) \xrightarrow{\sim} D^b_{\mathcal{B}_i}(\mathcal{B})$
  (cohomology in $\mathcal{A}_i$)

- $F[-p(i)]$ induces $\mathcal{A}_i/\mathcal{A}_{i-1} \simeq \mathcal{B}_i/\mathcal{B}_{i-1}$

$$\begin{tikzcd}
D^b(\mathcal{A})/D^b_{\mathcal{A}_{i-1}}(\mathcal{A}) \arrow[r, "{F[-p(i)]}"] & D^b(\mathcal{B})/D^b_{\mathcal{B}_{i-1}}(\mathcal{B}) \\
\mathcal{A}_i/\mathcal{A}_{i-1} \arrow[u, hook] \arrow[r, "\sim"] & \mathcal{B}_i/\mathcal{B}_{i-1} \arrow[u, hook]
\end{tikzcd}$$

The $\Theta : D^b(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{A})$ (in an $\mathfrak{sl}_2$-categorific-ation) is perverse w.r.t.
$\mathcal{A}_i = \{X \in \mathcal{A} \mid E^i X = 0\}$; $\mathcal{B}_i = \{X \in \mathcal{B} \mid F^i X = 0\}$
$p(i) = 1 - i$