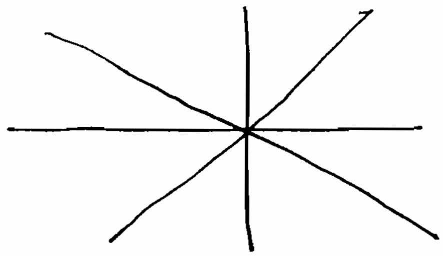
13 Jan 2009 — **Algebraic Lie Theory at the Newton Institute**

A. Ram — **Symmetry, polynomials and quantization I**



alcoves, root system type C

## Reflection group $W_0$

A *reflection* is $s \in GL_n(\mathbb{C})$ w/ exactly one eigenvalue not equal to 1.

$$s \text{ is conjugate to } \begin{pmatrix} \xi & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{pmatrix}$$

$\left.\begin{matrix} \mathfrak{h}_{\mathbb{Z}} \\ \mathfrak{h}^*_{\mathbb{Z}} \end{matrix}\right\}$ dual $\mathbb{Z}$-vector spaces

$$\langle \cdot , \cdot \rangle : \mathfrak{h}^*_{\mathbb{Z}} \times \mathfrak{h}_{\mathbb{Z}} \longrightarrow \tfrac{1}{e}\mathbb{Z}$$

where $e \in \mathbb{Z}_{>0}$

$W_0$ is a finite subgroup of $GL(\mathfrak{h}_{\mathbb{Z}})$ generated by $R^+$

$$R^+ = \{ s \in W_0 \mid s \text{ is a reflection} \}$$

$W_0$ acts on ~~the dua~~ $\mathfrak{h}_{\mathbb{Z}}^*$ by

$$\langle w\mu, \lambda^\vee\rangle = \langle \mu, w^{-1}\lambda^\vee\rangle, \quad w\in W_0,\ \lambda^\vee\in\mathfrak{h}_{\mathbb{Z}},\ \mu\in\mathfrak{h}_{\mathbb{Z}}^*$$

If $s\in R^+$ then fix $\alpha_s\in\mathfrak{h}_{\mathbb{Z}}^*$ and $\alpha_s^\vee\in\mathfrak{h}_{\mathbb{Z}}$

so that

$$s\mu = \mu - \langle\mu,\alpha_s^\vee\rangle\alpha_s \quad \text{and}$$

$$s^{-1}\lambda^\vee = \lambda^\vee - \langle\lambda^\vee,\alpha_s\rangle\alpha_s^\vee$$

<u>Double affine Weyl group</u>

$$\tilde{W} = \{ q^{k/e} X^\mu w Y^{\lambda^\vee} \mid k\in\mathbb{Z},\ \mu\in\mathfrak{h}_{\mathbb{Z}}^*,\ w\in W_0,\ \lambda^\vee\in\mathfrak{h}_{\mathbb{Z}} \}$$

w/ $q^{1/e}\in Z(\tilde{W})$ $\quad X^\mu X^\nu = X^{\mu+\nu},\quad Y^{\lambda^\vee}Y^{\sigma^\vee} = Y^{\lambda^\vee+\sigma^\vee}$

$$X^\mu Y^{\lambda^\vee} = q^{\langle\mu,\lambda^\vee\rangle} Y^{\lambda^\vee} X^\mu$$

$$wX^\mu = X^{w\mu}w \quad \text{and} \quad wY^{\lambda^\vee} = Y^{w\lambda^\vee}w$$

<u>The affine Weyl group</u>

$$W^\vee = \{ X^\mu w \mid \mu\in\mathfrak{h}_{\mathbb{Z}}^*,\ w\in W^0 \}$$

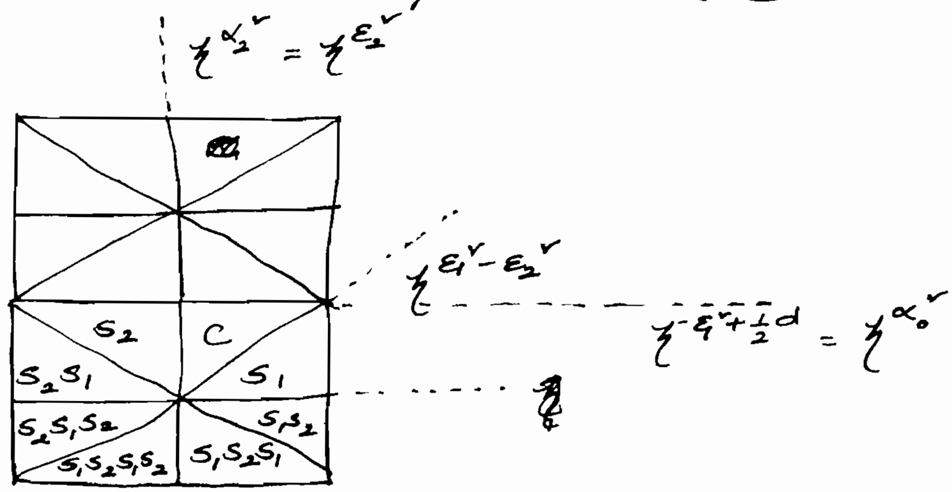
$W^\vee$ acts on $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}\otimes_{\mathbb{Z}}\mathfrak{h}_{\mathbb{Z}}^*$ by

$$X^\mu\nu = \mu+\nu, \quad \text{for } \mu\in\mathfrak{h}_{\mathbb{Z}}^*,\ \nu\in\mathfrak{h}_{\mathbb{R}}^*$$

The <u>alcoves</u> are the connected components of $\mathfrak{h}^*_{\mathbb{R}} \Big/ \bigcup_{\substack{\alpha^\vee \in R^+ \\ k \in \mathbb{Z}}} \mathfrak{h}^{\alpha^\vee + \frac{k}{e} d}$

where $\mathfrak{h}^{\alpha^\vee + \frac{k}{e} d} = \{ \nu \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \nu, \alpha^\vee \rangle = -k/e \}$

Fix an alcove $C$ w/ $0 \in \bar{C}$ (closure of $C$)




$\mathfrak{h}^{\alpha_0^\vee}, \mathfrak{h}^{\alpha_1^\vee}, \ldots, \mathfrak{h}^{\alpha_n^\vee}$ are the walls of $C$

$s_0^\vee, \ldots, s_n^\vee$ are the corresponding reflect-ions in $\mathfrak{h}^{\alpha_i^\vee}$.

$\Omega^\vee = \{ g^\vee \in W^\vee \mid g^\vee C = C \}$

$W_0 \xleftrightarrow{1-1} \left\{ \begin{array}{c} \text{alcoves in the} \\ 0 \text{ octagon} \end{array} \right\}$

$\mathfrak{h}^*_{\mathbb{Z}} = \{ \text{octagons} \}$

$$W^\vee \xleftrightarrow{1-1} \left\{ \text{alcoves in } \mathfrak{h}^* \Omega^\vee \times \mathfrak{h}^*_{\mathbb{R}} \right\}$$

$W^\vee$ is presented by generators $s_0^\vee, \ldots, s_n^\vee$ and $\Omega^\vee$ w/

$$\underbrace{s_i^\vee s_j^\vee \cdots}_{m_{ij}^\vee} = \underbrace{s_j^\vee s_i^\vee \cdots}_{m_{ij}^\vee} \quad \text{where } \frac{\pi}{m_{ij}^\vee} = \mathfrak{h}^{\alpha_i^\vee} \measuredangle\, \mathfrak{h}^{\alpha_j^\vee}$$

$$g^\vee s_i^\vee (g^\vee)^{-1} = s_{g(i)}^\vee \quad \text{where } g^\vee \mathfrak{h}^{\alpha_i^\vee} = \mathfrak{h}^{\alpha_{g(i)}^\vee}$$

$$(s_i^\vee)^2 = 1$$

<u>The affine braid group</u> $B^\vee$

generators: $T_0^\vee, \ldots, T_n^\vee$ and $\Omega^\vee$ w/

$$\underbrace{T_i^\vee T_j^\vee \cdots}_{m_{ij}^\vee} = \underbrace{T_j^\vee T_i^\vee \cdots}_{m_{ij}^\vee} \quad \text{and}$$

$$g^\vee T_i^\vee (g^\vee)^{-1} = T_{g(i)}^\vee$$

<u>Double affine braid group</u>

$W^\vee = \{ x^\mu w \mid \mu \in \mathfrak{h}^*_{\mathbb{Z}},\ w \in W_0 \}$ acts on

$\tilde{Y} = \{ q^{k/e} y^{\lambda^\vee} \mid k \in \mathbb{Z},\ \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}} \}$ by conjugation

Notation:

$$Y^{u\lambda^\vee} = u Y^{\lambda^\vee} u^{-1} \qquad u \in W^\vee, \lambda^\vee \in \mathfrak{h}_\#$$

Double affine braid group $\tilde{\tilde{B}}$ is generated by $\tilde{B}$ and $\tilde{Y}$ w/

$$q^{1/e} \in Z(\tilde{\tilde{B}}), \quad g^\vee Y^{\lambda^\vee} (g^\vee)^{-1} = Y^{g^\vee \lambda^\vee} \text{ for } g^\vee \in \Omega^\vee$$

$$(T_i^\vee)^{-1} Y^{\lambda^\vee} = \begin{cases} Y^{s_i^\vee \lambda^\vee} (T_i^\vee)^{-1}, & \text{if } \langle \lambda^\vee, \alpha_i \rangle = 0 \\ Y^{s_i^\vee \lambda^\vee} T_i^\vee & \text{if } \langle \lambda^\vee, \alpha_i^\vee \rangle = 1 \end{cases}$$

Double affine Hecke algebra $\tilde{\tilde{H}}$

Fix parameters
$t_i^{1/2}, \quad i = 0, \dots, n$

The double affine Hecke algebra $\tilde{\tilde{H}}$ is the quotient of $\mathbb{C}[\tilde{\tilde{B}}]$ by

$$(T_i^\vee)^2 = (t_i^{1/2} - t_i^{-1/2}) T_i^\vee + 1$$

Let $x^\mu = \hat{s}_{i_1}^\vee \dots s_{i_\ell}^\vee$ $g^\vee \leftarrow g^\vee \in \Omega^\vee$ be a minimal length walk to $x^\mu$ (in $W^\vee$)

The periodic orientation has
a) $C$ on +ve side of $\mathfrak{h}^{\alpha^\vee}$
b) $\mathfrak{h}^{\alpha^\vee}$ and $\mathfrak{h}^{\alpha^\vee + k/e\,d}$ have parallel orient-ations.

Define $X^\mu \in B^\vee$ by

$$X^\mu = \underbrace{g^\vee}_{g^\vee \in \Omega^\vee} (T_{i_1}^\vee)^{\epsilon_1^\vee} \cdots (T_{i_\ell}^\vee)^{\epsilon_\ell^\vee} \quad \text{where}$$

$$\epsilon_k^\vee = \begin{cases} -1 & \text{if } k^{th} \text{ step } \overset{-\,|\,+}{\longrightarrow} \\ +1 & \text{if } k^{th} \text{ step } \overset{-\,|\,+}{\longleftarrow} \end{cases}$$

* $X = \{X^\mu \mid \mu \in \mathfrak{h}^*_{\mathbb{Z}}\}$ ~~are~~ forms an abelian subgroup of $B^\vee$

$\tilde{H}$ has basis $\{q^{k/e} X^\mu T_w Y^{\lambda^\vee} \mid k \in \mathbb{Z}, \mu \in \mathfrak{h}^*_{\mathbb{Z}}, w \in W_0, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}\}$

where

$$T_w = T_{j_1}^\vee \cdots T_{j_\ell}^\vee \quad \text{if} \quad w = s_{j_1}^\vee \cdots s_{j_\ell}^\vee$$

is a minimal length walk to $w$.