14 Jan 2009 Algebraic Lie Theory at the Newton Institute

A. Ram Symmetry, polynomials and quantization II

Chevalley groups

Linear algebra Thm. 1 and 2

1) GL_n is generated by elementary matrices

$$x_{\varepsilon_i-\varepsilon_j}(f) = \begin{array}{cc} & j \\ i & \begin{pmatrix} 1 & & f \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \end{array}, \qquad f \in \mathbb{F}$$

$$h_{\varepsilon_i^\vee}(g) = \begin{array}{cc} i & \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & g & \\ & & & \ddots \\ 0 & & & & 1 \end{pmatrix} \end{array}, \qquad g \in \mathbb{F}^\times$$

2) $GL_n = \bigsqcup_{w \in W_0} B_0^+ w B_0^+$ where

$$B_0^+ = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \quad \text{and} \quad W_0 = S_n = \left\{ \begin{matrix}\text{permutati-} \\ \text{-on matri-} \\ \text{-ces}\end{matrix} \right\}$$

A Chevalley group G (over $\mathbb{F}$) is given by generators:

$$x_\alpha(f) \text{ and } x_{-\alpha}(f), \quad \alpha \in R^+, \; f \in \mathbb{F}$$

$$h_{\lambda^\vee}(g), \quad \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}, \; g \in \mathbb{F}^\times$$

w/ $T_0 = \langle h_{\lambda^\vee}(g) \,/\, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}},\ g \in \mathbb{F}^\times \rangle \cong \mathbb{F}^\times \times \mathbb{F}^\times \times \cdots \times \mathbb{F}^\times$

$W_0 = N/T_0$ (where N is the normalizer of T_0) is a finite group generated by R^+.

$\langle x_\alpha(f),\ x_{-\alpha}(f) \mid f \in \mathbb{F} \rangle \simeq SL_2$

$\mathbb{F} = \mathbb{C}((t))$ is the field of fractions of $\mathbb{C}[[t]] = \{ a_0 + a_1 t_1 + a_2 t^2 + \cdots \mid a_i \in \mathbb{C} \}$

$\mathbb{F} = \bigcup_{k \in \mathbb{Z}} t^k \mathbb{C}[[t]] = \{ a_k t^k + a_{k+1} t^{k+1} + \cdots \mid a_i \in \mathbb{C},\ k \in \mathbb{Z} \}$

$\mathbb{C}[[t]]^\times = \{ a_0 + a_1 t + a_2 t^2 + \cdots \mid a_i \in \mathbb{C},\ a_0 \in \mathbb{C}^\times \}$

Let

$$x_{\alpha + k\delta}(c) = x_\alpha(ct^k) \text{ and}$$

$$n_{\alpha+k\delta}(g) = x_{\alpha+k\delta}(g)\, x_{-\alpha-k\delta}(-g^{-1})\, x_{\alpha+k\delta}(g)$$

$W = W_0 \ltimes \mathfrak{h}_{\mathbb{Z}} = \{ w y^{\lambda^\vee} \,/\, w \in W_0,\ \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}} \}$ w/

$$y^{\lambda^\vee} y^{\sigma^\vee} = y^{\lambda^\vee + \sigma^\vee} \quad \text{and} \quad w y^{\lambda^\vee} = y^{w\lambda^\vee} w$$

T is the kernel of

$$\begin{aligned} N &\longrightarrow W \\ h_{\lambda^\vee}(t^{-1}) &\longmapsto y^{\lambda^\vee} \\ n_\alpha(1) &\longmapsto s_\alpha \end{aligned}$$

define

$$U_{\alpha,\geq k} = \{ x_\alpha(f) \mid f \in t^k \mathbb{C}[[t]] \}$$

$$U_\alpha = U_{\alpha \geq -\infty} = \{ x_\alpha(f) \mid f \in \mathbb{F} \}$$

and

$$U_0^- = \langle U_{-\alpha} \mid \alpha \in R^+ \rangle = \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \right\}$$

define subgroups

$$B_0^+ = \langle T_0, U_\alpha \mid \alpha \in R^+ \rangle = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$$

$$I = \left\langle \begin{matrix} T, U_{\alpha,\geq k}, \alpha \in R^+, k \in \mathbb{Z}_{\geq 0} \\ U_{-\alpha,\geq k}, \alpha \in R^+, k \in \mathbb{Z}_{>0} \end{matrix} \right\rangle$$

$$K = \left\langle \begin{matrix} T, U_{\alpha,\geq k}, \alpha \in R^+, k \in \mathbb{Z}_{\geq 0} \\ U_{-\alpha,\geq k}, \alpha \in R^+, k \in \mathbb{Z}_{\geq 0} \end{matrix} \right\rangle$$

G/B_0^+ is the flag <u>variety</u>

G/I is the <u>affine flag variety</u>

G/K is the <u>loop Grassmanian</u>

$G = \bigsqcup_{w \in W_0} B_0^+ w B_0^+$ (Bruhat decomposition)

$G = \bigsqcup_{w \in W} IwI$ and $G = \bigsqcup_{v \in W} U_0^- v I$

$$G = \bigsqcup_{\lambda^\vee \in \mathfrak{h}_{\#}/W_0} K h_{\lambda^\vee}(t^{-1}) K$$ (Cartan decompo-sition)

and

$$G = \bigsqcup_{\mu^\vee \in \mathfrak{h}_{\#}} U_0^- h_{\mu^\vee}(t^{-1}) K$$ (Iwasawa decomposi-tion)

<u>Hecke algebras = double coset algebras</u>

$\mathfrak{h}^{\alpha_0}, \ldots, \mathfrak{h}^{\alpha_n}$ are the walls of $C^\vee = 1$

$s_0, \ldots, s_n$ the corresponding reflections

$x_i(c) = x_{\alpha_i}(c)$ and $n_i = n_{\alpha_i}(1)$.

<u>Theorem</u> (Steinberg, Yale Lec. Notes)
Fix $w \in W$ and $w = s_{i_1} \cdots s_{i_\ell}$ a minimal length path to w

$$IwI = \{ x_{i_1}(c_1) n_{i_1}^{-1} \cdots x_{i_\ell}(c_\ell) n_{i_\ell}^{-1} I \mid c_1, \ldots, c_\ell \in \mathbb{C} \}$$

so that

$$IwI \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{labellings of the} \\ \text{walk } w = s_{i_1} \cdots s_{i_\ell} \end{array} \right\}$$

We prove this by induction by computing

$$IwI \cdot Is_jI = \{ x_{i_1}(c_1) n_{i_1}^{-1} \cdots x_{i_\ell}(c_\ell) n_{i_\ell}^{-1} \underbrace{I x_j(c) n_j^{-1} I}_{Is_jI} \}$$

<u>Case 1</u> $ws_j > w$ (ws_j is farther ~~away~~ from $C^\vee = 1$)

$$x_{i_1}(c_1) n_{i_1}^{-1} \cdots x_{i_\ell}(c_\ell) n_{i_\ell}^{-1} x_j(c) n_j^{-1} I \in I w s_j I$$

<u>Case 2</u> $\quad w s_j < w$ and $c = 0$

Then $w = s_{i_1} \cdots s_{i_{\ell-1}} s_j$

$$x_{i_1}(c_1) n_{i_1}^{-1} \cdots x_{i_{\ell-1}}(c_{\ell-1}) n_{i_{\ell-1}}^{-1} \underbrace{x_j(c_\ell) \underbrace{n_j^{-1} x_j(0) n_j^{-1}}_{\in I} I}_{\in I}$$

$$= x_{i_1}(c_1) n_{i_1}^{-1} \cdots x_{i_{\ell-1}}(c_{\ell-1}) n_{i_{\ell-1}}^{-1} I \in I w s_j I$$

<u>case 3</u> $\quad w s_j < w$ and $c \neq 0$

$$x_{i_1}(c_1) n_{i_1}^{-1} \cdots x_{i_{\ell-1}}(c_{\ell-1}) n_{i_{\ell-1}}^{-1} x_j(c_\ell) n_j^{-1} x_j(c) n_j^{-1} I$$

$$= x_{i_1}(c_1) n_{i_1}^{-1} \cdots x_{i_{\ell-1}}(c_{\ell-1}) n_{i_{\ell-1}}^{-1} x_j(c_\ell + c^{-1}) n_j^{-1} I \in I w I$$

Replace $\mathbb{C}$ w/ $\mathbb{F}_q$. Then

$$IwI \cdot I s_j I = \begin{cases} I w s_j I, & \text{if } w s_j > w \\ q\, I w s_j I + (q-1) I w I & \text{if } w s_j < w \end{cases}$$

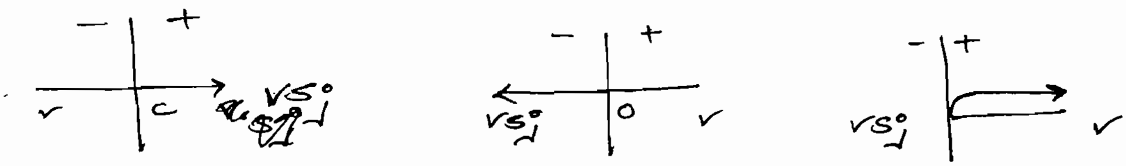
The double coset algebra has $T_0, \ldots, T_n$ (let $T_j = I s_j I$) w/

$$T_j^2 = (q-1) T_j + q$$

$$\underbrace{T_i T_j T_i \cdots}_{m_{ij}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij}} \quad \text{w/ } \pi / m_{ij} = \measuredangle\, \alpha_i^\vee \alpha_j^\vee$$

$$\text{Labeled } \underbrace{\text{positively folded alcove walks}}_{\text{Littelman path}}$$

A step of type/color j is



$c \in \mathbb{C}$ $\qquad\qquad\qquad\qquad$ $c \in \mathbb{C}^{\times}$

Theorem (Gaussent-Littelman, Parkinson-R-Schwer)

Fix $w, v \in W$ and $w = s_{i_1} \cdots s_{i_\ell}$ a minimal length walk to w.

$$IwI \cap U_0^- vI \xleftrightarrow{1-1} \left\{\begin{array}{l}\text{labeled positively} \\ \text{folded alcove walks} \\ \text{of type } i_1, \ldots, i_\ell \text{ that} \\ \text{end in } v\end{array}\right\}$$

Proof is by induction: compute

$$\left(IwI \cap U_0^- vI \right) \cdot I s_j I$$

$$x_{\sigma_1}(c_1') \cdots x_{\sigma_\ell}(c_\ell') n_v I \cdot x_j(c) n_j^{-1} I$$