15 Jan 2009 Algebraic Lie Theory at the Newton Institute

A. Ram Symmetry, polynomials and quantization III

$\left.\begin{array}{l}\mathfrak{h}_{\mathbb{Z}}\\ \mathfrak{h}^*_{\mathbb{Z}}\end{array}\right\}$ dual $\mathbb{Z}$-vector spaces

$$\langle\cdot,\cdot\rangle : \mathfrak{h}^*_{\mathbb{Z}} \times \mathfrak{h}_{\mathbb{Z}} \longrightarrow \tfrac{1}{e}\mathbb{Z}$$

The group algebras:

(*) $$K_T(pt) = \mathrm{span}\{x^\mu \mid \mu \in \mathfrak{h}^*_{\mathbb{Z}}\} \text{ w/ } x^\mu x^\nu = x^{\mu+\nu}$$
$$K_{T^\vee}(pt) = \mathrm{span}\{y^{\lambda^\vee} \mid \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}\} \text{ w/ } y^{\lambda^\vee} y^{\gamma^\vee} = y^{\lambda^\vee+\gamma^\vee}$$

$$K_T(pt) = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}],$$

$$K_{T^\vee}(pt) = \mathbb{C}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}]$$

$$x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n} \quad \text{if } \mu = \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n$$

$$y^{\lambda^\vee} = y_1^{\lambda_1} \cdots y_n^{\lambda_n} \quad \text{if } \lambda^\vee = \lambda_1\varepsilon_1^\vee + \cdots + \lambda_n\varepsilon_n^\vee$$

where $\{\varepsilon_i\}$, $\{\varepsilon_i^\vee\}$ are bases for $\mathfrak{h}^*_{\mathbb{Z}}$ and $\mathfrak{h}_{\mathbb{Z}}$ respectively.

Let $q^{\frac{1}{e}}$ be a parameter, $q^{\frac{1}{e}} \in Z(\mathcal{Q})$

The Heisenberg group is

$$\mathcal{Q} = \{q^{k/e} x^\mu y^{\lambda^\vee} \mid k \in \mathbb{Z}, \mu \in \mathfrak{h}^*_{\mathbb{Z}}, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}\}$$

w/ (*) and $x^\mu y^{\lambda^\vee} = q^{\langle \mu, \lambda^\vee\rangle} y^{\lambda^\vee} x^\mu$

$\left.\begin{array}{l}\mathfrak{h}_{\mathbb{C}}\\ \mathfrak{h}^*_{\mathbb{C}}\end{array}\right\}$ dual vector spaces $\langle \cdot , \cdot \rangle : \mathfrak{h}^*_{\mathbb{C}} \times \mathfrak{h}_{\mathbb{C}} \longrightarrow \mathbb{C}$

The <u>symmetric</u> <u>algebras</u>

$$S(\mathfrak{h}^*_{\mathbb{C}}) = H_T(pt) = \mathbb{C}[x_1, \ldots, x_n]$$

$$S(\mathfrak{h}_{\mathbb{C}}) = H_{T^\vee}(pt) = \mathbb{C}[\partial_1, \ldots, \partial_n]$$

w/ $x_\mu = \mu_1 x_1 + \cdots + \mu_n x_n$ if $\mu = \mu_1 \varepsilon_1 + \cdots + \mu_n \varepsilon_n$

$\partial_{\lambda^\vee} = \lambda_1 \partial_1 + \cdots + \lambda_n \partial_n$ if $\lambda^\vee = \lambda_1 \varepsilon_1^\vee + \cdots + \lambda_n \varepsilon_n^\vee$

Let κ be a parameter. The <u>Weyl algebra</u> $\mathcal{D}$ is generated by

$\mathbb{C}[x_1, \ldots, x_n]$ and $\mathbb{C}[\partial_1, \ldots, \partial_n]$

w/ $\partial_{\lambda^\vee} x_\mu = x_\mu \partial_{\lambda^\vee} + \kappa \langle \mu, \lambda^\vee \rangle$

$\mathcal{D}$ acts on polynomials: If $\langle \varepsilon_i^\vee, \varepsilon_j \rangle = \delta_{ij}$, $\kappa = 1$

$$\partial_i = \frac{\partial}{\partial x_i}, \quad \left[\frac{\partial}{\partial x_j}, x_i\right] = \frac{\partial}{\partial x_j} x_i - x_i \frac{\partial}{\partial x_j} = \delta_{ij} = \langle \varepsilon_i, \varepsilon_j^\vee \rangle$$

W_0 is a finite subgroup of $GL(\mathfrak{h}_{\mathbb{C}})$ generated by

$R^+ = \{ s \in W_0 \mid s \text{ is a reflection} \}$

The group algebra is

$\mathbb{C}[W_0] = \text{span}\{ t_w \mid w \in W_0 \}$ w/ $t_{w_1} t_{w_2} = t_{w_1 w_2}$

W_0 acts on $\mathfrak{h}^*_{\mathbb{C}}$ by $\langle w\mu, \lambda^\vee \rangle = \langle \mu, w^{-1} \lambda^\vee \rangle$

For each $s \in R^+$ fix $\alpha_s \in \mathfrak{h}^*_{\mathbb{C}}$ and $\alpha_s^\vee \in \mathfrak{h}_{\mathbb{C}}$
so that $s\mu = \mu - \langle \mu, \alpha_s^\vee \rangle \alpha_s$, ~~and~~
$s^{-1}\lambda^\vee = \lambda^\vee - \langle \lambda^\vee, \alpha_s \rangle \alpha_s^\vee$,
$\alpha_{wsw^{-1}} = w\alpha_s$ and $\alpha^\vee_{wsw^{-1}} = w\alpha_s^\vee$ for $w \in W_0$

Introduce parameters:

$$c_s, \ s \in R^+, \qquad c_s = c_{wsw^{-1}} \text{ for } w \in W_0$$

The <u>rational</u> <u>Cherednik</u> <u>algebra</u> $\tilde{H}$ is generated by
$\mathbb{C}[x_1, \dots, x_n]$, $\mathbb{C}[D_1, \dots, D_n]$ and $\mathbb{C}[W_0]$
w/

$$t_w x_\mu = x_{w\mu} t_w, \qquad t_w D_{\lambda^\vee} = D_{w\lambda^\vee} t_w$$

$$D_{\lambda^\vee} x_\mu = x_\mu D_{\lambda^\vee} + \kappa \langle \mu, \lambda^\vee \rangle - \sum_{s \in R^+} c_s \langle \alpha_s, \lambda^\vee \rangle \langle \mu, \alpha_s^\vee \rangle t_s$$

As a vector space:

$$\tilde{H} \cong \mathbb{C}[x_1, \dots, x_n] \otimes \mathbb{C}[W] \otimes \mathbb{C}[D_1, \dots, D_n]$$

If $p \in \mathbb{C}[x_1, \dots, x_n]$ then

$$D_{\lambda^\vee} p = p D_\lambda + \kappa (d_{\lambda^\vee} p) - \sum_{s \in R^+} c_s \langle \alpha_s, \lambda^\vee \rangle (\Delta_s p) t_s$$

where $d_{\lambda^\vee} : \mathbb{C}[x_1, \dots, x_n] \longrightarrow \mathbb{C}[x_1, \dots, x_n]$
given by $d_{\lambda^\vee}(x_\mu) = \langle \mu, \lambda^\vee \rangle$,
$d_{\lambda^\vee}(p_1 p_2) = p_1 d_{\lambda^\vee}(p_2) + d_{\lambda^\vee}(p_1) p_2$

and $\Delta_s : H_T(pt) \longrightarrow H_T(pt)$

$$p \longmapsto \frac{p - sp}{x_{\alpha_s}} \quad \text{(BGG-operator)}$$

The subalgebra $\mathbb{H}$ given by $\mathbb{C}[D_1, \ldots, D_n]$ and $\mathbb{C}[W_0]$ has a 1-dim module span$\{\mathbb{1}\}$ given by

$$t_w \mathbb{1} = \mathbb{1} \quad \text{and} \quad D_{\lambda^\vee}\mathbb{1} = 0$$

The <u>polynomial representation</u> of $\tilde{\mathbb{H}}$ is

$$\mathrm{Ind}_{\mathbb{H}}^{\tilde{\mathbb{H}}} \mathbb{1} = \tilde{\mathbb{H}}\mathbb{1} = \mathbb{C}[x_1, \ldots, x_n]\mathbb{1}$$

$D_{\lambda^\vee}$ acts on $\tilde{\mathbb{H}}\mathbb{1}$ by the <u>Dunkl operator</u>

$$D_{\lambda^\vee} = \kappa d_{\lambda^\vee} - \sum_{s \in R^+} c_s \langle \lambda^\vee, \alpha_s \rangle \frac{1}{x_{\alpha_s}} (1-s)$$

suppose W_0 is a finite subgroup of $GL(\mathfrak{h}_{\mathbb{R}})$ generated by R^+. Another presentation of W_0 has generators $s_1, \ldots, s_n$ and

$$s_i^2 = 1, \quad \underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}}$$

where $\pi / m_{ij} = \mathfrak{h}^{\alpha_i} \measuredangle \mathfrak{h}^{\alpha_j}$.

Let

$\mathbb{C}[y_1, \ldots, y_n] \simeq S(\mathfrak{h}_{\mathbb{C}})$ w/ $y_{\lambda^\vee} = \lambda_1 y_1 + \cdots + \lambda_n y_n$

if $\lambda^\vee = \lambda_1 \varepsilon_1^\vee + \cdots + \lambda_n \varepsilon_n^\vee$

The <u>trigonometric Cherednik algebra</u> $\tilde{H}_{gr}$ is generated by

$\mathbb{C}[y_1, \ldots, y_n]$, $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $\mathbb{C}[W_0]$

w/

$$t_w X^\mu = X^{w\mu} t_w,$$

$$t_{s_i} y_{\lambda^\vee} = y_{s_i\lambda^\vee} t_{s_i} + c_{s_i} \langle \lambda^\vee, \alpha_i \rangle \quad \text{for } i = 1, \ldots, n$$

and

$$y_{\lambda^\vee} X^\mu = X^\mu y_{\lambda^\vee} + \kappa \langle \mu, \lambda^\vee \rangle X^\mu - \sum c_s \langle \lambda^\vee, \alpha_s \rangle \frac{X^\mu - X^{s\mu}}{\ldots} t_s$$

The subalgebra H_{gr} generated by $\mathbb{C}[Y_1, \ldots, Y_n]$ and $\mathbb{C}[W_0]$ has a 1-dim module span $\{\mathbb{1}\}$ given by
$t_w \mathbb{1} = \mathbb{1}$, $Y_{\lambda^\vee} \mathbb{1} = \langle \rho_c, \lambda^\vee \rangle \mathbb{1}$, where
$\langle \rho_c, \alpha_i^\vee \rangle = c_{s_i}$ for $i = 1, \ldots, n$

The <u>polynomial representation</u> of $\tilde{H}_{gr}$

$$\mathrm{Ind}_{H_{gr}}^{\tilde{H}_{gr}} \mathbb{1} = \tilde{H}_{gr} \mathbb{1} = \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \mathbb{1}$$
$$= K_T(pt) \mathbb{1}$$

The operator $\pi_s : K_T(pt) \longrightarrow K_T(pt)$
$\pi_s X^\mu = \dfrac{X^\mu - X^{s\mu}}{1 - X^\alpha}$ is the <u>Demazure operator</u>

$Y_{\lambda^\vee}$ acts on $\tilde{H}_{gr} \mathbb{1}$ by the <u>Dunkl-Cherednik operator</u>

$$Y_{\lambda^\vee} = \langle \rho_c, \lambda^\vee \rangle + \kappa \partial_{\lambda^\vee} - \sum_{s \in R^+} c_s \langle \lambda^\vee, \alpha_s \rangle \frac{1}{1 - X^\alpha} (1 - s)$$

where

$$\partial_{\lambda^\vee} : \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \longrightarrow \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$$
$$X^\mu \longmapsto \langle \mu, \lambda^\vee \rangle X^\mu$$

<u>"Isomorphisms"</u>

$$H_T(pt) = \mathbb{C}[x_1, \ldots, x_n] \xrightarrow{ch} K_T(pt) = \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$$

$$e^{x_\mu} \longleftarrow X^\mu$$

Let

$$X^\mu = e^{h x_\mu} \qquad \left(\in \tilde{H}[[h]] \right)$$

~~Φ_{λ^r}~~ Then $\partial_{\lambda^r} = \frac{1}{h} d_{\lambda^r}$

$$\tilde{\mathbb{H}}[[h]]^{reg} \xrightarrow{ch} \tilde{H}_{gr}$$

$$e^{hx_\mu} \longleftarrow\!\!\mapsto x^\mu$$

$$\frac{1}{h} d_{\lambda^r} \longleftarrow\!\!\mapsto \partial_{\lambda^r}$$

$$t_\omega \longleftarrow\!\!\mapsto t_\omega$$

$$\langle \rho_c, \lambda^r \rangle + \frac{1}{h} D_{\lambda^r} + \sum_{s \in R^+} c_s \langle \lambda^r, \alpha_s \rangle \left(\frac{1}{x_{\alpha_s}} - \frac{1}{1 - e^{h x_{\alpha_s}}} \right)(1 - t_s) \longleftarrow\!\!\mapsto Y_{\lambda^r}$$