19 Jan 2009 — **Algebraic Lie Theory at the Newton Institute**

R. Rouquier — **Higher Representations of Lie Algebras I**

Representation theory: groups/algebras acting on vector spaces

(take K_0) $\uparrow$ decategorification

2-Representation theory: monoidal categories acting on categories (abelian, triangulated)

$\mathcal{A}$ - monoidal category.

$$\mathcal{A} \xrightarrow[\substack{\text{monoidal} \\ \text{(exact)}}]{} \mathrm{Fun}(\mathcal{V},\mathcal{V})$$

abelian

A = monoid ~~or~~ w/ objects $\mathrm{Ob}(\mathcal{A})/_{\simeq}$

$V = \mathbb{C} \otimes K_0(\mathcal{V})$, get $A \longrightarrow \mathrm{End}_{\mathbb{C}}(V)$ (a representation of A)

There are examples of monoidal categories with interesting 2-rep. theory.

$\mathfrak{g}$ = ~~Kacs~~ Kac-Moody algebra
$\rightsquigarrow \mathcal{A}(\mathfrak{g})$ - monoidal category.

eg. $\mathfrak{g} = \mathfrak{sl}_2$ (cf. J. Chuang's lectures)

1) Hecke algebras

k - Noetherian commutative ring
(may as well assume $k = \mathbb{C}$)

I - (finite) set
← not really required

Let $Q = Q_{i,j}(u,v)_{i,j \in I}$ be a matrix

w/ $Q_{i,j}(u,v) \in k[u,v]$ ~~&~~ ~~Q_{ii}~~ s.t
$Q_{ii} = 0$

Given $n \geq 1$, define a k-algebra $H_n(Q)$ by generators and relations (as a subalgebra of End (polynomial ring)

~~Let $L = I^n$~~ Put $L = I^n$

Generators

$*\ 1_\nu, \quad \nu \in I^n$
$*\ x_{a,\nu}, \quad 1 \leq a \leq n, \ \nu \in I^n$
$*\ T_{a,\nu}, \quad 1 \leq a \leq n-1, \ \nu \in I^n$

Relations: $*\ 1_\nu 1_{\nu'} = \delta_{\nu\nu'} 1_\nu$ (idempotents: 1_ν)

$*\ 1_{\nu'} x_{a,\nu} 1_{\nu''} = \delta_{\nu,\nu'} \delta_{\nu,\nu''} x_{a,\nu}$
(loops: $x_{a,\nu}$)

$*\ 1_{\nu'} T_{a,\nu} 1_{\nu''} = \delta_{\nu,\nu''} \delta_{s_a(\nu),\nu'} T_{a,\nu}$

(edges: $T_{a,\nu}$)

S_n acts on I^n, w/ $s_a = (a, a+1)$

quiver: ~~xx~~ vertex set I^n

$\nu \circlearrowleft x_{a,\nu}$

$\nu \; \overset{T_{a,\nu}}{\text{———}} \; s_a(\nu)$

$*\quad x'_{a,\nu} \cdot x_{b,\nu} = x_{b,\nu} \cdot x_{a,\nu}$

$*\quad T_{a,\nu} \cdot x_{b,\nu} - x_{s_a(b),\, s_a(\nu)}\, T_{a,\nu}$

$$= \begin{cases} -1_\nu & \text{if } a=b \text{ and } \nu_a = \nu_{a+1} \\ 1_\nu & \text{if } b = a+1 \text{ and } \nu_a = \nu_{a+1} \\ 0 & \text{otherwise} \end{cases}$$

$*\quad \cancel{T_{s_a(\nu),\nu} \cdot T}\; T_{a,s_a(\nu)} \cdot T_{a,\nu} = Q_{\nu_a,\nu_{a+1}}(x_{a,\nu}, x_{a+1,\nu})$
(quadratic relation)

$*\quad T_{a,s_b(\nu)} \cdot T_{b,\nu} = T_{b,s_a(\nu)}\, T_{a,\nu}$ if $|a-b| > 1$
(elements commute if far enough)

$*\quad T_{a+1} T_a T_{a+1} \cdot 1_\nu - T_a T_{a+1} T_a 1_\nu$

$$= \begin{cases} (x_{a+2} - x_a)^{-1}\left(Q_{\nu_a,\nu_{a+1}}(x_{a+2}, x_{a+1}) - Q_{\nu_a,\nu_{a+1}}(x_a, x_{a+1})\right) & \text{if } \nu_a = \nu_{a+2} \\ 0 & \text{if } \nu_{a+2} \neq \nu_a \end{cases}$$

$\cancel{\nu_{a+2} \neq \nu_a}$

(braid relations)

eg: $|I| = 1$, $Q = 0$

$H_n(Q) =$ nil affine Hecke algebra
$= \langle x_1, \ldots, x_n, T_1, \ldots, T_{n-1} \mid \text{relations} \rangle$

vary the polynomial, ~~how~~ Q, how does the algebra behave?

Generating family for $H_n(Q)$ as a k-module:
given $w \in S_n$, choose $(i_1, \ldots, i_\ell) \in \{1, \ldots, n-1\}^\ell$
such that $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced decomposition of w.

$$\mathcal{Y} = \{(i_1, \ldots, i_\ell)\}_{w \in S_n}$$

$$\mathcal{B} = \{T_{i_1} \cdots T_{i_\ell} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \cdot 1_\nu\}_{\substack{\nu \in I^n \\ \alpha_1, \ldots, \alpha_n \geq 0 \\ (i_1, \ldots, i_\ell) \in \mathcal{Y}}}$$

here $x_a = \sum_{\nu' \in I^n} x_{a,\nu'}$

<u>Prop</u> B is a basis
$\iff Q_{ij}(u,v) = Q_{ji}(v,u) \quad \forall i,j$

so restricting to such Q gives a flat family of algebras

$$R = \bigoplus_{\nu \in I^n} k[x_1, \dots, x_n] 1_\nu$$

Action of $H_n(Q)$ on R ("essentially" a quiver representation):

* 1_ν: id on ν-component, 0 elsewhere
* $x_{i,\nu}$: x_i on ν-component, 0 elsewhere

Let $P = (P_{ij}(u,v))_{i,j \in I}$, $P_{ii} = 0$, s.t

$$Q_{ij}(u,v) = P_{ij}(u,v) P_{ji}(v,u)$$

* $\underbrace{T_{a,\nu} : R_\nu \to R_{s_a(\nu)}}$ given by $(x_a - x_{a+1})^{-1}(s_a - 1)$ if $s_a(\nu) = \nu$

0 elsewhere $\qquad P_{\nu_a, \nu_{a+1}}(x_{a+1}, x_a) \cdot s_a$ otherwise

<u>Prop</u> R is a faithful representation of $H_n(Q)$.

(This is ~~essentially~~ like $\mathrm{Ind}_{H_n^{\mathrm{fin}}}^{H_n^{\mathrm{aff}}} \mathbb{1}$ for the "usual" Hecke algebra)

~~These algo~~ Specializations of these algebras show up in the geometry of quiver varieties

$A = (a_{ij})_{i,j\in I}$ symmetric Cartan matrix $\begin{cases} a_{ii} = 2 \\ a_{ij} \in \mathbb{Z}_{\leq 0} & \text{if } i \neq j \end{cases}$

Define Q associated w/ A.

$k = \mathbb{Z}[\{t_{ij}^{\pm 1}\}, \{t_{i,j,r,s}\}]$

~~Let m~~ $m_{ij} = -a_{ij}$; &

$i \neq j$ $\quad Q_{ij}(u,v) = t_{ij} u^{m_{ij}} + t_{ji} v^{m_{ij}} + \sum_{r,s \leq m_{ij}} t_{ijrs} u^r v^s$

$(m_{ij} \neq 0)$.

w/ $t_{ijrs} = t_{jisr}$, $t_{ij} = t_{ji}$ (when $m_{ij} = 0$)

if $m_{ij} = 0$: $Q_{ij}(u,v) = t_{ij}$

$\rightsquigarrow$ get $H_n(A) = H_n(Q)$

<u>Note</u>: Cartan matrix $\approx$ graph, vertex set I, m_{ij} edge $i - j$ $(i \neq j)$.

Given an orientation of that graph define

$Q_{ij} = (-1)^{\#\{i \to j\}} (u-v)^{m_{ij}}$ get the

algebra $H_n(\Gamma)$, Γ – the given quiver.

$H_n(\Gamma)$ is graded, $\deg x_a = 2$, $\deg \tau_{a,\nu} = m_{\nu_a, \nu_{a+1}}$