20 Jan 2009 — Algebraic Lie Theory at the Newton Institute

R. Rouquier — Higher Representation Theory of Lie Algebras II

$A=(a_{ij})_{i,j\in I}$ symmetric Cartan matrix.
$\mathcal{B}(A)$ monoidal category ~~over~~ enriched over $R=\mathbb{Z}[\{t_{ij}^{\pm 1}\},\{t_{ijrs}\} \mid i\neq j\in I,\ 0\le r,s<m_{ij}]$

Generators of $\mathcal{B}(A)$,

- objects E_i, $i\in I$.

i.e objects of $\mathcal{B}(A)$ are finite sums of $\otimes$ product of E_i's.

- arrows $x_i : E_i \longrightarrow E_i$

$T_{ij} : E_iE_j \longrightarrow E_jE_i$

(Note: $E_iE_j = E_i\otimes E_j$)

Relations

- $T_{ji}T_{ij} = Q_{ji}(E_ix_j,\ x_iE_j)$

$$\left(\text{Recall: } Q_{ij}(u,v) = t_{ij}u^{m_{ij}} + t_{ji}v^{m_{ij}} + \sum t_{ijrs}u^r v^s \qquad (m_{ij}\neq 0)\right)$$

- $T_{ij}\circ(x_iE_j) - (E_jx_i)T_{ij} = \delta_{ij}$

- $T_{ij}\circ(E_ix_j) - (x_jE_i)T_{ij} = -\delta_{ij}$

- In $\mathrm{Hom}(E_iE_jE_k,\ E_kE_jE_i)$,

$$(T_{jk}E_i)\circ(E_jT_{ik})\circ(T_{ij}E_k) - (E_kT_{ij})\circ(T_{ik}E_j)\circ(E_iT_{jk}) = \underbrace{(x_iE_jE_i - E_iE_jx_i)}^{-1}\big(Q_{ij}(x_iE_j,\ E_ix_j)E_i - E_iQ_{ij}(E_jx_i,\ x_jE_i)\big)$$

(poly divides one to its right) if $i=k$

and 0 if $i\neq k$

$n \geq 1$

$$H_n(A) \xrightarrow{\sim} \mathrm{End}\left(\bigoplus_{\nu \in I^n} E_{\nu_n} \otimes \cdots \otimes E_{\nu_1}\right)$$

$$1_\nu \longmapsto \text{projection onto } \nu \text{ term}$$

$$x_{a,\nu} \longmapsto E_{\nu_n} \otimes \cdots \otimes E_{\nu_{a+1}} \otimes x_{\nu_a} E_{\nu_{a-1}} \otimes \cdots \otimes E_{\nu_1}$$

$$T_{a,\nu} \longmapsto E_{\nu_n} \cdots E_{\nu_{a+2}} T_{\nu_{a+1},\nu_a} E_{\nu_{a-1}} \cdots E_{\nu_1}$$

<u>divided powers</u> want to define $\frac{E^n}{n!}$.

so want $E^n \simeq \underbrace{E^{(n)} \oplus E^{(n)} \oplus \cdots \oplus E^{(n)}}_{n! \text{ times}}$

~~so find the~~ i.e $E^n \simeq n!\, E^{(n)}$. want to find the right idempotent in $\mathrm{End}(E^n)$

claim:

$$H_n(\{1\}) \xrightarrow{\sim} \mathrm{End}(E_i^n)$$

$$\|$$

${}^0H_n$: nil affine Hecke algebra

$$\|$$

$$\mathbb{Z}[x_1, \ldots, x_n] \otimes {}^0H_n^{fin}$$

${}^0H_n^{fin} = \mathbb{Z}\langle T_1, \ldots, T_{n-1}\rangle/_\sim$: nil Hecke algebra $(T_i^2 = 0)$

$w \in S_n$ say $w = s_{r_1} \cdots s_{r_d}$ is a reduced word, set

$$T_w = T_{r_1} \cdots T_{r_d} \quad (\text{independent of choice of reduced word})$$

$\{T_w\}_{w \in S_n}$ is a basis of ${}^0H_n^{fin}$

$$b_n = T_{w_0} : x_1^{n-1} x_2^{n-2} \cdots x_{n-1} \in {}^0H_n$$

w_0 = largest element of S_n ; $b_n^2 = b_n$

$$\underbrace{b_n\,{}^0H_n}_{\wr} \xrightarrow[\text{multi-plication}]{\sim} {}^0H_n$$

$$\mathbb{Z}[x_1,\dots,x_n]\otimes \mathbb{Z}[x_1,\dots,x_n]^{S_n}$$

($b_n\,{}^0H_n$ gives a Morita equivalence:
$\mathbb{Z}[x_1,\dots,x_n]^{S_n} \longleftrightarrow {}^0H_n$)

If M is a ${}^0H_n$-module then $n_0!(b_n M)\simeq M$

<u>Defn</u> $E_i^{(n)} = b_n E_i^n \in \mathcal{B}^{idem}(A)$

$\mathcal{B}^{idem}(A)$ is the <u>idempotent completion</u> of $\mathcal{B}(A)$;
$\mathcal{B}(A)$ is a full subcategory of $\mathcal{B}^{idem}(A)$;
objects of $\mathcal{B}^{idem}(A)$ = direct summands of obje-cts of $\mathcal{B}$.

<u>Kac-Moody algebra associated w/ A</u>

algebra generated by e_i, f_i, h_i $(i\in I)$
relations: $[h_i,h_j]=0$, $[e_i,h_j]=a_{ij}e_i$,
$[f_i,h_j]=-a_{ij}f_i$

$$\mathrm{ad}(e_i)^{1-a_{ij}}(e_j)=0\ ;\quad \mathrm{ad}(f_i)^{1-a_{ij}}(f_j)=0$$

$(\mathrm{ad}(x)(y)=[x,y])$; $[e_i,f_j]=\delta_{ij}h_i$

$U_{\mathbb{Z}}$ = $\mathbb{Z}$-subalgebra generated by

$$e_i^{(n)}=\frac{e_i^n}{n!}\ ;\ f_i^{(n)}=\frac{f_i^n}{n!}\ ,\ h_i\ ,\ i\in I\ ,\ n\geq 1$$

$U^+_{\mathbb{Z}}$ - $\mathbb{Z}$-subalgebra generate by $e_i^{(n)}$

<u>Prop</u>: Given $i \neq j \in I$, $m = m_{ij} = -a_{ij}$

$$\bigoplus_{r\text{-even}} E_i^{(m-r+1)} E_j E_i^{(r)} \cong \bigoplus_{r\text{ odd}} E_i^{(m-r+1)} E_j E_i^{(r)}$$

canonical

<u>Cor</u> There exists an algebra map

$$U^+_{\mathbb{Z}} \longrightarrow K_0(\mathcal{B}^{\text{idem}})$$

$$e_i^{(n)} \longmapsto [E_i^{(n)}]$$

$$K_0(\mathcal{B}^{\text{idem}}) = \bigoplus_{M \in \mathcal{B}^i/\sim} \mathbb{Z}[M] \Big/ [M] = [M_1] + [M_2] \text{ if } M \simeq M_1 \oplus M_2$$

recall: not assuming this category to be abelian

Assume A comes from an oriented quiver. This gives $k \longrightarrow \mathbb{Z}$:

$$\mathcal{B}^{\text{idem}}_{\mathbb{Z}} \supset \mathcal{B}_{\mathbb{Z}} = \mathcal{B} \otimes_k \mathbb{Z}$$

can be graded: i.e Hom spaces are $\mathbb{Z}$-graded: $\deg x_i = 2$, $\deg T_{ij} = a_{ij}$

Associated graded category $\mathcal{B}^{\bullet}_{\mathbb{Z}}, (\mathcal{B}^{\text{idem}}_{\mathbb{Z}})^{\bullet}$ i.e <u>objects</u> can be graded. (by the root lattice ?? or length of root lattice element?)

$K_0((\mathcal{B}^{\text{idem}}_{\mathbb{Z}})^{\bullet})$ is a $\mathbb{Z}[v, v^{-1}]$-module where $v[M] = [M[1]]$

<u>Prop</u> There is an algebra map

$$U^+_{\mathbb{Z}[v,v^{-1}]} \longrightarrow K_0\left(\left((\mathcal{B}_{\mathbb{Z}})^{\text{idem}}\right)^{\bullet}\right)$$

$\uparrow$

quantum group.

(The map is in fact an isomorphism)

canonical basis $\rightsquigarrow$ indecomposable projectives

Get all of $U_{\mathbb{Z}[v,v^{-1}]}$ by the Drinfeld double method.

How do you recover the Hopf algebra structure? (WHOLE REASON TO PLAY THIS GAME).

Have to pass to A_∞-categories to do this.