21 Jan 2009 **Algebraic Lie Theory at the Newton Institute**

R. Rouquier **Higher representations of Lie algebras III**

$A = (a_{ij})_{i,j\in I}$ symmetric Cartan matrix.
(Assume I is finite)
consider a root datum of this type:
X, Y free abelian groups, $\langle \cdot,\cdot\rangle : Y \times X \longrightarrow \mathbb{Z}$
a perfect pairing. $I \longrightarrow X$, $I \longrightarrow Y$ w/
$i \longmapsto \alpha_i$, $i \longmapsto \alpha_i^\vee$
linearly independent images and s.t. $\langle \alpha_i^\vee, \alpha_j\rangle = a_{ij}$.
There is an associated Lie algebra $\mathfrak{g} = \langle e_i, f_i, h_s \mid i\in I, s\in Y\rangle$.

Define a category $\mathcal{W}(\mathfrak{g})$, $\mathbb{C}$-linear,
objects $= X$; arrows via generators $e_i : \lambda \longmapsto \lambda+\alpha_i$
$(\lambda\in X)$, $f_i : \lambda \longmapsto \lambda - \alpha_i$ and relations

- $e_i f_j = f_j e_i$ if $i \neq j$.
- $e_i f_i = f_i e_i + \langle \alpha^\vee, \lambda\rangle \, id_\lambda$

  End(λ)

- $\displaystyle\sum_{a+b=1-a_{ij}} (-1)^a e_i^{(a)} e_j e_i^{(b)} = 0$; analogously w/ f_i

  $i\neq j$

Let $R : \mathcal{W}(\mathfrak{g}) \longrightarrow \mathbb{C}\text{-mod}$ be a $\mathbb{C}$-linear functor
then $V = \bigoplus_{\lambda\in X} R(\lambda)$ is a rep. of $\mathcal{W}(\mathfrak{g})$.

This gives an equivalence $\boxed{\text{not script } W}$
Functors$(\mathcal{W}(\mathfrak{g}), \mathbb{C}\text{-mod}) \xrightarrow{\sim} \left(\begin{array}{c}U(\mathfrak{g})\text{-modules w/} \\ \text{wt. space decom-} \\ \text{position}\end{array}\right)$

<u>Remark</u> ~~$\mathbb{Z}$-version~~ There is a $\mathbb{Z}$-version and a quantum version.

<u>Define $\mathcal{U}$ - a 2-category:</u>

$Ob(\mathcal{U}) = X$

<u>Recall</u>: a 2-category has arrows and 2-arrows, i.e a category enriched in categories (Hom's are categories).

<u>example</u> the 2-category of all categories (say small).

$\Bbbk = \mathbb{Z}[\{t_{ij}^{\pm 1}\}, \{t_{ij\,rs}\}]$

$\mathcal{B}$ = monoidal category $/\Bbbk$ generated by $E_i, i \in I$

$x_i : E_i \longrightarrow E_i$, $T_{ij} : E_i E_j \longrightarrow E_j E_i$ + relations.

Define $\mathcal{B}_1$: $\Bbbk$-linear (strict), monoidal category obtained from $\mathcal{B}$ by adding F_i: right duals to E_i, $i \in I$. So we throw in

$\varepsilon_i : E_i F_i \longrightarrow id$

$\eta_i : id \longrightarrow F_i E_i$ w/ usual properties of units and counits.

we want to enforce "$e_i f_j = f_j e_i$, ..."

Let L = subcategory of $\mathcal{B}_1$, w/ objects products of E_is and F_is

$h : Ob(L) \longrightarrow X$ (morphism of monoids)

$E_i \longmapsto \alpha_i$

$F_i \longmapsto -\alpha_i$

Given $\beta \in X$, put $\mathcal{B}_1(\beta)$ = full subcategory (additive) of $\mathcal{B}_1$ w/ objects direct sums of objects of $h^{-1}(\beta)$

Define $\mathcal{U}_1$: 2-category; objects $= X$,
$\mathrm{Hom}(\lambda, \lambda') = \mathcal{B}_1(\lambda' - \lambda)$

~~Define~~ $\cancel{\sigma_{ij}: E_iE_j \longrightarrow E_jE_i}$ ~~via~~

$\cancel{E_iE_j \longrightarrow E_jE_i}$

Define $\sigma_{ij} : E_iF_j \longrightarrow F_jE_i$ via.

$$E_iF_j \xrightarrow{\eta_j E_i F_j} F_jE_jE_iF_j \xrightarrow{F_j T_{ji} F_j} F_jE_iE_jF_j \xrightarrow{F_jE_i\varepsilon_j} F_jE_i$$

(this is the same as.

$$X \otimes Y \longrightarrow Y \otimes Y^* \otimes X \otimes Y \longrightarrow Y \otimes X \otimes Y^* \otimes Y \longrightarrow Y \otimes X$$
).

if $\langle \alpha_i^\vee, \lambda \rangle \geq 0$, define (in $\mathcal{U}_1$)

$$\rho_{i,\lambda} : E_iF_i \cdot 1_\lambda \longrightarrow F_iE_i 1_\lambda \oplus 1_\lambda^{\langle \alpha_i^\vee, \lambda\rangle} \quad (\text{in } \mathrm{End}(\lambda))$$
$(E_iF_i : \lambda \to \lambda)$ as $\sigma_{ii} + \sum_{r=0}^{\langle \alpha_i^\vee, \lambda\rangle - 1} \varepsilon_i \circ (x_i^r F_i)$

$\mathcal{U} = \mathcal{U}_1[\{\sigma_{ij}^{-1}\}_{i \neq j}, \{\rho_{i,\lambda}^{-1}\}]$ is the 2-category associated w/ $\mathfrak{g}$.

0-arrows : X
1-arrows : sums of products of E_i, F_i's
2-arrows : generated by $x_i, T_{ij}, \varepsilon_i, \eta_i, \sigma_{ij}^{-1}, \rho_{i,\lambda}^{-1}$

Put $\mathcal{U}^{idem}$ = idempotent completion of $\mathcal{U}$ (add direct summands of products of E_is, F_is).

Let $\mathcal{C}$ be the additive category w/ objects X, $\mathrm{Hom}(\lambda,\lambda') = K_0(\mathcal{H}om_{\mathcal{U}^{idem}}(\lambda,\lambda'))$.

<u>Prop</u> There is an ~~isomorphism~~ of a functor ~~of~~

$$\begin{aligned} \dot{\mathcal{U}}_{\mathbb{Z}}(\mathfrak{g}) &\longrightarrow \mathcal{C} \\ \lambda &\longmapsto \lambda \\ e_i^{(n)} &\longmapsto [E_i^{(n)}] \\ f_j^{(n)} &\longmapsto [F_j^{(n)}] \end{aligned}$$

which induces

$$\dot{\mathcal{U}}_{\mathbb{Z}}(\mathfrak{g}) \xrightarrow{\sim} \mathcal{C}_{\mathbb{Z}}$$

where $\mathcal{C}_{\mathbb{Z}}$ is defined ~~by~~ ~~...~~ using $\mathcal{U} \otimes_{\mathbb{k}} \mathbb{Z}$

As F_i is right dual to E_i: $\mathrm{End}(F_i^{n}) \xrightarrow{\sim} \mathrm{End}(E_i^{n})^{op}$

$$b_n$$

$F_i^{(n)} = b_n \cdot F^n$

(In practice for reps (on categories) of this 2-cat $\mathcal{U}$ it is hard to check invertibility. In next lecture we will see that it is enough to check $E_iF_j = F_jE_i$ at the level of K_0)

"2-representations of $\mathcal{U}$"
= 2-functor

$$\begin{aligned} \mathcal{U} \longrightarrow\ & \text{2-cat of } \mathbb{k}\text{-linear cats} \\ \text{or } & \text{2-cat of } \text{ " abelian cats.} \\ \text{or } & \text{2-cat of } \text{ " } \Delta\mathcal{G}\text{-cats.} \end{aligned}$$