22 Jan 2009 Algebraic Lie Theory at the Newton Institute

R. Rouquier Higher Representations of Lie algebras IV

$\mathcal{A}$: 2-category over $\mathbb{k}$. $Ob(\mathcal{A}) = X$; 1-arrows = sum of products of E_is & F_is.
2-arrows = $\mathbb{k}$-mod

associated to a cartan matrix A

$\{\mathcal{V}_\lambda\}_{\lambda\in X}$ $\mathbb{k}$-linear categories. A 2-rep of $\mathcal{A}$ on $\{\mathcal{V}_\lambda\}_{\lambda\in X}$ (or on $\oplus \mathcal{V}_\lambda$) is a 2-functor:

$$\mathcal{A} \xrightarrow{R} \text{2-cat of } \mathbb{k}\text{-linear categories}$$

w/ $R(\lambda) = \mathcal{V}_\lambda$.

concretely ? data of:

- functors $E_i : \mathcal{V}_\lambda \longrightarrow \mathcal{V}_{\lambda+\alpha}$
  $F_i : \mathcal{V}_\lambda \longrightarrow \mathcal{V}_{\lambda-\alpha}$

- morphisms $x_i : E_i \longrightarrow E_i$
  $T_{ij} : E_iE_j \longrightarrow E_jE_i$
  $\left.\begin{matrix}\varepsilon_i\\ \eta_i\end{matrix}\right\}$ adjunctions (E_i, F_i)

subject to the conditions:
- x_i, T_{ij} satisfy the Hecke relations from Lecture 1
- $\sigma_{ij}, \rho_{i,\lambda}$ are invertible $i \neq j$.

Def 2-rep is integrable if E_i, F_i's are locally nilpotent. if for $M \in \mathcal{V}_\lambda$, $E_i^n M = 0$ for $n \gg 0$}.

<u>Type A</u> k-field, $I \subset k$ (finite)

quiver Γ_I w/ vertices: I; arrows $i \to i+1$ if $i, i+1 \in I$.

Types of connected components:

A_n, $\tilde{A}_{p-1}$ (char $k = p$) $\rightsquigarrow \mathfrak{sl}_I$

<u>Def</u> An $\mathfrak{sl}_I$-categorification (given $\mathcal{V}$ a k-linear abelian category. Assume $\mathrm{End}(L) = k$ for L-simple, objects have finite composition series) is the data:

1) ~~An~~ Adjoint pair (E, F) of exact functors $\mathcal{V} \to \mathcal{V}$

2) $X \in \mathrm{End}(E)$, $T \in \mathrm{End}(E^2)$

3) $\mathcal{V} = \bigoplus_{\lambda \in X} \mathcal{V}_\lambda$ a decomposition of $\mathcal{V}$

s.t.

if E_i = generalized i-eigenspace of X on E ($i \in k$), $/\!/$y for F_i

then 1) $E = \bigoplus_{i \in I} E_i$

2) $e_i = [E_i]$, $f_i = [F_i]$ define an integrable action of $\mathfrak{sl}_I$ on $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{V})$

3) $F \simeq$ left adjoint of E.

4) degenerate affine Hecke relations

$$\bar{H}_n \longrightarrow \mathrm{End}(E^n) \qquad X_i \longmapsto E^{n-i} X_i E^{i}$$
$$T_i \longmapsto E^{n-i-1} T E^{i-1}$$

5) $E_i \mathcal{V}_\lambda \subset \mathcal{V}_{\lambda + \alpha_i}$; $F_i \mathcal{V}_\lambda \subset \mathcal{V}_{\lambda - \alpha_i}$

RR

<u>Thm</u> $\{\mathfrak{sl}_I$ - categorification on $V\}$

$$\wr\wr$$

$\{$ (exact) 2-rep of $\mathcal{U}_k$ on $V\}$ integrable

$$\mathcal{U}_k = \mathcal{U} \otimes_{\mathbb{k}} k$$

<u>proof</u> • step 1: deg. affine Hecke algebra of S_n $\simeq$ Hecke algebra of Γ_I

• step 2: 2-rep def: why adjunction (F_i, E_i)?

• step 3: 2-cat: why σ_{ij}, $\rho_{s,\lambda}$ are isomorphisms?

<u>construct "simple" 2-representations</u>

$X^+ = \{\lambda \in X \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in R^+\}$

"1 rep": $\Delta(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g})^+} \mathbb{C}_\lambda \twoheadrightarrow L(\lambda)$

unique simple quotient

$\{L(\lambda)\}_{\lambda \in X^+}$ = simple integrable reps of $U(\mathfrak{g})$

<u>Fix λ:</u>

$V_\mu = \mathrm{Hom}_{\mathcal{U}}(\lambda, \mu)$, $\mu \in X$

$\{V_\mu\}$: 2-rep of $\mathcal{U}$

$E_i : V_\mu \longrightarrow V_{\mu + \alpha_i}$

$\mathrm{Hom}(\lambda, \mu)$ $\qquad$ $\mathrm{Hom}_{\mathcal{U}}(\lambda, \mu + \alpha_i)$

$$\mathcal{L} : \lambda \to \mu \longmapsto E \circ \mathcal{L} : \lambda \to \mu + \alpha_i$$
(NOT INTEGRABLE)

$\mathcal{V}'_\mu$ = full subcategory of $\mathrm{Hom}(\lambda, \mu)$ w/ objects direct sums of $\mathcal{L} \cdot E_i$, $\mathcal{L} : \lambda + \alpha_i \to \mu$

$$\mathcal{L}(\lambda)_\mu := \mathcal{V}_\mu / \mathcal{V}'_\mu$$

Prop $L(\lambda) \simeq \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{L}(\lambda))$ as a $U(\mathfrak{g})$-rep

$$\left(L(\lambda) = \Delta(\lambda) \Big/ \left(\sum_i U(\mathfrak{g})\, e_i^{\langle \lambda, \alpha_i^\vee \rangle + 1} v_\lambda^+ \right) \right)$$

proof ~~of~~ (prop) $\mathrm{End}_{\mathcal{A}} (E_i^{\langle \lambda, \alpha_i^\vee \rangle + 1}) \simeq \mathrm{Hom}$

$$\mathrm{Hom}\left(E_i^{\langle \lambda, \alpha_i^\vee \rangle}, F_i E_i^{\langle \alpha_i^\vee, \lambda \rangle + 1} \right)$$

Thm $\mathcal{V}$ a 2-rep of $\mathcal{A}$, $\mathcal{V} = \mathcal{V}^{\mathrm{idem}}$
Assume every object of $\mathcal{V}$ is in $\mathcal{A} v_\lambda^+$.
Then,
$$\mathcal{L}(\lambda) \otimes_{Z(\mathcal{L}(\lambda)_\lambda)} \mathcal{V}_\lambda \xrightarrow{\sim} \mathcal{V}$$

Thm $\mathcal{V}$ 2-rep, integrable $\mathcal{V} = \mathcal{V}^{\mathrm{idem}}$.
assume
$\{\lambda \mid \mathcal{V}_\lambda \neq 0\}$ bounded. Then there exists a filtration
$$0 = \mathcal{V}\{0\} \subset \mathcal{V}\{1\} \subset \cdots \subset \mathcal{V}\{n\} \subset \mathcal{V}$$
s.t $\mathcal{V}\{l\} / \mathcal{V}\{l-1\}$ is λ_l generated for some $\lambda \in X^+$.